The Shift from Classical to Modern Probability: a Historical Study with Didactical and Epistemological Reflexions

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Abstract

The Shift from Classical to Modern Probability: a Historical Study with Didactical and Epistemological Reflexions

Vinicius Gontijo Lauar

In this thesis, we describe the historical shift from the classical to the modern definition of probability. We present the key ideas and insights in that process, from the first definition of Bernoulli, to Kolmogorov's modern foundations discussing some of the limitations of the old approach and the efforts of many mathematicians to achieve a satisfactory definition of probability. For our study, we've looked, as much as possible, at original sources and provided detailed proofs of some important results that the authors have written in an abbreviated style.

We then use this historical results to investigate the conceptualization of probability proposed and fostered by undergraduate and graduate probability textbooks through their theoretical discourse and proposed exercises. Our findings show that, despite textbooks give an axiomatic definition of probability, the main aspects of the modern approach are overshadowed by other content. Undergraduate books may be stimulating the development of classical probability with many exercises using proportional reasoning while graduate books concentrate the exercises on other mathematical contents such as measure and set theory without necessarily proposing a reflection on the modern conceptualization of probability.

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Chapter 1

Introduction

1.1 The scope of the thesis

This thesis was catalyzed by two curiosities we had in mind: if probability has been studied for many centuries, i) why do its foundations date from 1933? and ii) why is it associated to measure theory[1]?

The foundations of the modern theory of probability were laid out by the Russian mathematician Andreï Nikolaïevitch Kolmogorov in his book: *Foudations of the Theory of Probability*[2] in 1933. At the beginning of the 18^{th} century, Jacques Bernoulli and Abraham de Moivre published the first works with the definition of probability that became, two centuries later, following the work of Kolmogorov, a generalized and abstract theory of probability. Although Cardano, Montmort, Pascal and others had already made advances with the calculations of the number of possible outcomes for two or three die throws and the addition and multiplication rules, the outstanding breakthrough of Bernoulli and de Moivre, in relation to their predecessors, is that they were the first to define probability and expectation with a greater level of generality. Bernoulli discovered and proved the first version of a very important convergence theorem, the law of large numbers, and de Moivre was aware of the generality of the results that others were applying to specific problems.

The classical definition of probability by Bernoulli and de Moivre remained essentially the

[1] Measure theory started with the works of Borel and Lebesgue in the transition from the 19^{th} to the 20^{th} century.
[2] The origial version was written in German and is called "Grundbegriffe der Wahrscheinlichkeitsrechnung"

same throughout the 18^{th} and 19^{th} centuries. Yet, as science evolved through time, contradictions and paradoxical results began to reveal the limitations of classical probability, requiring a new and precise definition of probability and other related concepts. It was not until the development of measure theory and the Lebesgue integral beyond Euclidean spaces that the modern and axiomatic definition of probability in its complete and abstract form was developed and probability was raised from a set of tools in applied mathematics to a branch on its own.

Kolmogorov's modern definition of probability may be seen by an unaware and naive person as a fully-born concept. A sudden, brilliant and original idea that triumphed over chaos and confusion. Even if Kolmogorov's book contains some original contributions, it is also seen as a work of synthesis [56]. History of mathematics cannot be limited to the formal results presented in the standard textbooks. The imprecise and contradictory developments also play an important role in the advances of science [19], [17]. The advances in mathematics are almost always built on the work of people who contribute little by little over hundreds of years. Eventually, someone is able to distinguish the valuable ideas of their predecessors among the myriad of statements to fit existing knowledge into a new approach. This was exactly the case of Kolmogorov, because many results from measure theory[3], set theory[4], probability[5] and even unsuccessful attempts to an axiomatization,[6] were relevant to his foundation of modern probability. The famous statement attributed to Newton: "*If I have seen further it is by standing on the shoulders of Giants*" also applies to Kolmogorov.

Given the above context, we can state the first problem this thesis sought to answer: If probability has been present in mathematics for many centuries, why the advent of measure theory was a turning point in the definition and conceptualization of probability. More specifically, why did probability *need* measure theory as its basis to be considered an autonomous branch of mathematics?

By understanding this evolution from classical to modern probability and the importance of Kolmogorov's axiomatization up to the point that probability was raised to an autonomous branch

[3]Example of authors: Borel, Lebesgue, Carathéodory, Fréchet, Radon and Nikodyn
[4]Example of authors: Cantor and Hausdorff
[5]Example of authors: Borel, Cantelli, Lévy, Slutsky and Steinhaus
[6]Example of authors: Laemmel, Hilbert, Broggi, Lomnicki, Bernstein, von Mises and Slutsky

of mathematics, a second research question that attaches a didactical value to this thesis emerged: Considering the classical and the modern approaches to probability, which one of them are primarily advanced by undergraduate and graduate textbooks?

The investigation of this problem started in a literature review on epistemological obstacles in mathematics and in probability. Due to the near absence of studies considering probability at the post-secondary level, we've done a pilot test. We've interviewed four graduate students to investigate whether their conceptualization of probability is closer to a classical or to a modern approach. We have found a persistence of two epistemological obstacles[7]. The first one is the obstacle of equiprobability, that is, a tendency to believe that elementary events are *equiprobable* (i.e., uniform) by nature. The second is the proportionality obstacle or illusion of linearity, that is, the epistemological obstacle of using proportional reasoning in situations where it is not appropriate. In probability the illusion of linearity comes from the habit of identifying probability as a ratio of favourable over possible cases.

Once the obstacles of equiprobability and proportionality were found in the pilot test, we've looked at some undergratuate and graduate textbooks used in the four universities in Montreal with the goal of identifying how those books introduce the definition of probability and how they help students develop, though the exercises and examples, a modern or a classical view of the domain.

1.2 Context and originality of our study

In the previous section we've presented the context regarding the shift in probability from a classical approach to a modern theory developed by Kolmogorov. In this thesis, this evolution of probability is detailed with some relevant mathematical results developed in full, based on original sources[8], to evidence some mathematical ideas or to present in details some proofs. The goal is to display great ideas from each author's contribution to the foundations of modern probability.

We tried, as much as possible, to bring attention to the motivation for the discoveries and also to present some ideas that were unsuccessful to show that the development of the theory didn't

[7] See chapter 2 for more details.

[8] When the original source was available in English or French.

follow a straightforward path.

The didactical contribution is also original, because there is a scarcity of research in post-secondary level of probability learning. While the proportionality obstacle has been identified as a common epistemological obstacle in high school, we have identified its persistence in graduate level studies. The obstacle of equiprobability has been researched in different educational levels, but here we apply it along with the illusion of linearity to the conceptualization of probability. Furthermore, we've also done an investigation into the approach to probability taken by the books used most commonly by Montreal universities, as well as an anaylysis of the proposed exercises, seeking to find some of the potential sources for the proportionality and equiprobability obstacles. These didactical reflections appeal to readers interested in the teaching of probability at the undergraduate and graduate levels.

1.3 The outline of the thesis

In the second chapter we present a literature review on three important topics for this thesis: i) epistemological obstacles in mathematics education, ii) examples of epistemological obstacles in probability and iii) misconceptions in probability.

The third chapter presents a pilot study with graduate students aimed at discovering whether these students conceptualize probability in a classical or in a modern sense, or using a mix of both. We've found that the epistemological obstacle of identifying probability as a ratio of favourable over possible cases in situations where it doesn't apply is persistent and we associate it to the obstacles of proportionality and equiprobability.

The fourth chapter answers by itself one of the main goals of this thesis. It explain why probability became attached to measure theory at the beginning of the 20^{th} century. More specifically, it explains why probability *needed* measure theory as its basis to be considered an autonomous branch of mathematics. The chapter presents the origins of probability and its development, including the first definition of this concept and the contributions from Bernoulli and de Moivre. It also discusses the evolution of measure theory with focus on the results that were important to the development of modern probability, and the association of both disciplines since its foundation

with Borel and Lebesgue. We also expose the need to develop a general and abstract set of axioms for probability and the first attempts at an axiomatization. At the end of the chapter, we discuss Borel's denumerable probability, more specifically the use of countable additivity and the strong law of large numbers, two essential results to the foundation of the axioms.

In the fifth chapter we discuss the axiomatic definition of probability in Kolmogorov's book for finite and infinite spaces and the change in the concept of conditional probability to illustrate how this step into modern probability established a fertile ground to rigorous and general definitions of terms that were loosely used in the classical era. As an illustration, there is an example that leads to a paradox in classical probability that was resolved by Kolmogorov's new approach using conditional probability.

The sixth chapter analyzes some of the most commonly used probability textbooks in the four universities in Montreal. The goal is to analyze how those books introduce the definition of probability and if their proposed exercise sets require students thinking about Kolmogorov's innovation or if they stimulate the idea of probability as a ratio of favourable over possible cases – even perhaps reinforcing the epistemological obstacles of equiprobability and proportionality.

The thesis finishes with the seventh chapter, where we draw a summary of the findings and discuss some recommendations for the teaching of probability.

Chapter 2

Literature Review

2.1 Introduction

This literature review is focused on epistemological obstacles and other sources of difficulties in learning probability. The focus is the concept of epistemological obstacles in learning mathematics, some examples of epistemological obstacles in probability and a brief survey of some difficulties in learning probability. This literature review doesn't concern all the difficulties in learning probability and doesn't intend to cover the whole subject of epistemological obstacles, but rather to present the definition of the term and exemplify how it applies to probability, besides showing some common difficulties in probability that have been studied.

The literature on epistemological obstacles and difficulties in learning probability is very extensive, however, we didn't find any publications related to the teaching and learning of probability at the post-secondary level, and in particular, no publications related to the teaching and learning of the axiomatic definition of probability. This is exactly the gap that this thesis aims to contribute to fill up.

This review is presented in four sections. After this introduction, the second section discusses texts that introduce the concept of epistemological obstacles in mathematics. The third section applies this concept to probability and gives three examples. The fourth section presents some of the research in difficulties in learning probability and the chapter finishes with some closing remarks.

2.2 What are epistemological obstacles?

Brousseau [15] and [16] was the first to transpose the concept of epistemological obstacle to the didactics of mathematics by highlighting the change that the theory of epistemological obstacles proposes in the status of the *error*: *L'erreur n'est pas seulement l'effet de l'ignorance, de l'incertitude, [...] mais l'effet d'une connaissance antérieure, qui avait son intérêt, ses succès, mais qui, maintenant, se révèle fausse, ou simplement inadaptée*" [16] (p. 104).

The term epistemological obstacle was proposed by Bachelard [3] in his studies of the history and philosophy of science. The concept was first applied to mathematics education by Brousseau [15], [16]. Among the learning obstacles, Brousseau distinguishes three categories: i) ontogenic obstacles, genetic and psychological obstacles developed as a result of the cognitive and personal development of the student, ii) didactic obstacles, which come from the didactic choices and iii) the epistemological obstacles, that happen because of the nature of the mathematical concepts themselves and from which there is no escape due to the fact that they play a constitutive role in the construction of knowledge.

In this review, we will focus on epistemological obstacles, because we are interested in the obstacles related to the nature of the mathematical concepts, such as probability, random variable and mathematical expectation. Sierpinska [62] defined epistemological obstacles as "*ways of understanding based on some unconscious, culturally acquired schemes of thought and unquestioned beliefs about the nature of mathematics and fundamental categories such as number, space, cause, chance, infinity,... inadequate with respect to the present day theory*" (p. xi).

As an example, the daily life usage of the word *limit* as a barrier that should not be crossed may be an epistemological obstacle that the student needs to confront when studying the limit of a function. Similarly, the vast experience acquired with linearity from early school years and many daily life situations often leads to an inclination to use linear models or a proportional reasoning where these should not be applied. As an example, many people think that getting 2 heads out of three coin tosses is equally likely to 6 heads in nine coin tosses.

Sierpinska [61] and [62], concerned with mathematical learning, describes the concept of understanding as an act involved in a process of interpretation. This interpretation process is the

development of a dialectic between more and more elaborate guesses and validations of these guesses. With this interpretation of understanding, she describes the relationship between epistemological obstacles and understanding in mathematics.

At a certain moment, typically when facing a new problem, we discover that our current mathematical knowledge is not accurate (e.g., understanding limit as a barrier may be accurate in the context of finding the limits of rational functions - horizontal asymptotes - but is no longer accurate when studying limits of functions that oscillate about their limit). This is when we become aware of an epistemological obstacle. So we understand something and we start knowing in a new way, which might turn into another epistemological obstacle in another situation. The *act of understanding* is the act of overcoming an epistemological obstacle. Sierpinska points out that some acts of understanding may turn out as acquiring new epistemological obstacles.

In many cases overcoming an epistemological obstacle and understanding are just two ways of speaking about the same thing. Epistemological obstacles look backwards, focusing the attention on what was wrong, insufficient, in our ways of knowing. Understanding looks forward to the new ways of knowing. We do not know what is really going on in the head of a student at the crucial moment but if we take the perspective of his or her past knowledge we see him or her overcoming an obstacle, and if we take the perspective of the future knowledge, we see him or her understanding.

2.2.1 The role of non-routine tasks in facing epistemological obstacles

As Sierpinska [62] explains, the successive acts of understandings are obtained through facing rather than avoiding epistemological obstacles. Hardy [32] and Schoenfeld [55], among others, discuss the role of tasks, typically given to students in *hiding* epistemological obstacles. Hardy studied how college students learn calculus and more specifically, the influence of routine tasks and the institutional environment in their way of thinking and solving problems. Most of the tasks that students face (thus called *routine tasks*) when they learn limits are of the type *find the limit* of a continuous function or of a function whose required limit becomes trivial after some common algebraic operations.

To Schoenfeld, each group of routine tasks adds a mathematical tool kit to the student and the

sum of these techniques reflects the corpus of mathematics that the student should learn. This environment of blocs of routine tasks enhances the view of mathematics as a canon, instead of a science. As consequences of routine tasks: i) students are not expected to figure out the methods by themselves and acquire a passive behaviour because they think that the only valid method to solve a given set of problems is the one provided by the instructor; ii) it also makes students think that one should have a ready method for the solution of the mathematical problems and iii) generates an automatic behaviour towards tasks, as the students read the first few words of a problem, they already know what will be asked and what is the method that should be used. Practices based on routine tasks and weak theoretical content do not challenge students' modes of thinking, in particular, they don't *force* students to face epistemological obstacles. Both authors, show and illustrate how non-routine tasks, carefully crafted to reveal misconceptions, make students confront and overcome them, thus advancing their learning.

For students to gain a sense of the mathematical enterprise, their experience with mathematics must be consistent with the way mathematics is done. The artificiality of the examples moves the corpus of exercises from the realm of the practical and plausible to the realm of the artificial, which makes students give up to make sense of mathematics. Sierpniska, Schoenfeld and others emphasize that the focus should be changed from content to modes of thinking. Handling new and unfamiliar tasks, possibly using unknown methods should be at the heart of problem solving. While routine tasks may foster a passive behaviour, non-routine tasks, if well elaborated, can help students to confront their epistemological obstacles and promote successive acts of understandings.

2.3 Examples of epistemological obstacles in probability

In the previous section we introduced the term epistemological obstacle in mathematics. In this section we present some examples of those obstacles in probability. As will be shown in chapter five, those obstacles played a significant role in the evolution of the theory of probability as they consisted of granted beliefs about *chance* that lead to theoretical inconsistencies and difficulties in solving problems.

2.3.1 The obstacle of determinism

Borovcnik and Kapadia [14] describe probability from a historical an philosophical perspective. According to them, since the roman empire, when Christianity became the only allowed religion under Theodosius (around 380 A.D.), games of chance, which were a great incentive to the development of probability, lost prestige as everything that happens is determined by the will of god. The dominant idea was that randomness comes from man's ignorance instead of the nature of the events. This belief that any phenomena is deterministic and could be predicted with absolute certainty if we were aware of all the variables of influence is what we call the obstacle of determinism. This epistemological obstacle has existed from ancient times, passing through the classical era of probability and is still present in some people's mind today. In the original texts of Bernoulli [7], DeMoivre [21] and Laplace [41], we can see that, as it was common during their time, they considered the world to be deterministic. The omnipotent and omniscient god determines every event, usually by causal laws, leaving no space to chance. Hence probability, was a tool used to make decision due to our ignorance of all the factors that determine an event [30]. Von Plato [67] shows the reluctance in accepting randomness in the essence of matter in the early 20^{th} century.

2.3.2 The obstacle of equiprobability

The obstacle of equiprobability came from the idea that elementary events are equiprobable. Laplace created the *principle of indifference*, where he attributed equal probability to all events when we have no reason to suspect that any one of the cases is more likely to occur than the others. This principle was adopted in his definition of probability: "*La théorie des hasards consiste à réduire tous les événements du même genre, à un certain nombre des cas également possibles, c'est-à-dire, tels que nous soyons également indécis sur leur existence; et à déterminer le nombre de cas favorables à l'événement dont on cherche la probabilité. Le rapport de ce nombre à celui de tous les cas possibles, est la mesure de cette probabilité qui n'est ainsi qu'une fraction dont le numérateur est le nombre des cas favorables, et dont le dénominateur est le nombre de tous les cas possibles*" [41] (p. iv). The principle of indifference and the definition of probability as a ratio of equally likely cases have shown its limitations in probability and that are one of the main motivations for the foundations

of modern probability, as we describe in chapter four.

The obstacle of equiprobability is introduced in the literature by [45]. Gauvrit and Morsanyi [25] describe it as the tendency of using a uniform distribution for events where it is not appropriate. They argue that many times, although not always, this obstacle is present because some experiments consist in analyzing a non-uniform random variable that was originated by the combination of two or more uniform random variables. Among many examples in modern literature involving this obstacle, we will present the two children problem and the Monty Hall problem.

In the two children problem, consider a person has two kids. If at least one of them is a boy, what is the probability that both children are boys? The correct answer is easily found by setting equally likely ordered pairs: (girl, boy), (boy, girl) and (boy, boy), so the correct answer would be 1/3. However, when unordered pairs are considered, {girl, boy} or {boy, boy}, they are not equally likely. Their probabilities are, respectivelly, 2/3 and 1/3, but many people consider that they share the same probability of the ordered pairs, so give the incorrect answer of 0.5.

Another example of the equiprobability obstacle is the very well known Monty Hall problem. In a game, a participant should choose one out of three doors, say A, B or C. Behind one of them, there is a prize and behind the others, there isn't. The participant picks a door, say C. After that, one door without the prize is opened (say B) and is shown to the participant. Then it is asked to the participant if she/he would prefer to stay with door C or to change to door A. Thinking that doors A and C have the same probability of having the prize after door B opened, is an incorrect reasoning given by the obstacle of equiprobability. At the first moment, all three doors have the same probability of having the prize. Once the participant has picked door C, the probability that the prize is in the set $A \cup B$ is 2/3. When presenter of the game opens door B, it is not done at random, because she/he knows that the prize is not in door B. That means that the door A has probability 2/3 of having the prize while door C has probability 1/3. So the best strategy would be to change doors.

The interpretation of equiprobability of elementary events is problematic, specially when the probability space, Ω, is infinite (countable or not). In a countable infinite space, by countably additivity, $P(\Omega)$ must be either 0, if each elementary event has probability zero, or infinity, if to each elementary event would be assigned one constant positive probability.

In an uncountable probability space, like the interval $[0, 1]$, let's consider any sub-interval $(a, b) \subset [0, 1]$. If we set $P\{x \in (a, b)\} = b - a$, that is, the probability of x be in the sub-interval (a, b) is the length of that interval, then we say that x is uniformly chosen at random. Intuition may suggest that if we provide two descriptions of one set of elementary outcomes that can be bijectively related to each other, then if in one of them the elementary outcomes are equiprobable, the same should be true under the other description. However, this epistemological obstacle leads to a paradox found in Poincaré [51] and in Borel [12]. Let $y = x^2$. Can x and y be considered uniformly chosen at random? For any $x \in [0, 1]$, we find a corresponding $y \in [0, 1]$. The probability that $x \in [0, 1/2]$ is 1/2, and the probability that $y \in [0, 1/4]$ is 1/4, but both probabilities should be the same, according to the bijection established between the descriptions of the elements of the interval $[0, 1]$.

2.3.3 The obstacle of proportionality or illusion of linearity

The proportionality obstacle or illusion of linearity is "*...the strong tendency to apply linear or proportional models anywhere, even in situations where they are not applicable*" [65] (p. 113). The illusion of linearity is classified as an epistemological obstacle because it has implications in the historical development of probability, but it is also considered a didactic obstacle due to the extensive attention given to proportional reasoning in mathematics education.

The illusion of linearity takes place because the notions of proportion and chance are cognitively and intuitively very closely related to each other. The over-reliance in proportions cause *errors* in probability thinking. The classical definition of probability, as we will discuss in chapter four, is given by a fraction or proportion of favourable over possible cases. Thus, comparing probabilities is a comparison of two fractions, so proportional reasoning is considered to be a basic tool in this domain since the first notions of chance in the 16^{th} and 17^{th} centuries even before the classical definition.

The obstacle of proportionality may be found in terms of *distance* between two probability values, specially when we consider events of probability 0 or probability 1. Let's take a non-symmetric coin, with probability of heads $p \neq 0.5$. We toss the coin repeatedly many times and register the relative frequency of heads. The law of large numbers tells us that the difference

between the relative frequency of heads in that sequence and the value 0.5 could be made arbitrarily small by making p arbitrarily close to 0.5. This situation doesn't apply when we consider a coin with probability of heads arbitrarily close (but not equal) to 0 and another coin with probability of heads equal to zero.

To see that, let's take two biased coins, the first with probability of showing heads of 0.0001 and the second probability 0.00001. Even if the difference $|0.0001 - 0.00001|$ is very small, the ratio $0.001/0.0001$ makes the expected number of heads in the first n outcomes 10 times greater in the first sequence than in the second one. A coin α with any arbitrarily small, but positive, probability of heads produces infinitely different sequences than a coin β with probability of heads equal to zero. This happens because the coin α should produce sequences of outcomes with infinitely many heads and the coin β should show a finite number of it, which configures a very different behaviour.

The Italian mathematician Cardano (1501–1576) made considerable gains in gambling because of his knowledge of chance[1]. He correctly reasoned that the probability of getting double ones in a die throw is 1/36, but felt into the obstacle of proportionality when he thought that he had to throw the dice 18 times to have a probability of 1/2 to get a double ones at least once ($18 \times 1/36 = 1/2$).

De Méré (1607-1684), a notorious gambler, knew by experience that it was advantageous to bet on at least 1 six in 4 rolls of a die. Using a proportional reasoning, he thought that it was also advantageous to bet on at least 1 double-six in 24 rolls of two fair dice (4/6 = 24/36). It was Pascal who explained him that the probability of 1 six in 4 trials equals $1 - (5/6)4 = 0.52$, but 1 double-six in 24 rolls of two dice is $1 - (35/36)24 = 0.49$.

The illusion of linearity is also reiforced by didactical choices, which makes it a didactical obstacle. In this sense, one of the causes of the illusion of linearity is the extensive attention given to proportional reasoning in mathematics education. As the proportional (or linear) model is a key concept in primary and secondary education with a very wide applicability, students get so familiar with it that they usually stick to a linear approach in situations where it doesn't apply.

In fact, Piaget and Inhelder [49] believe that understanding proportions and ratios is essential for children to understand probability. Lamprianou and Afantiti Lamprianou [40] suggest that

[1] In Cardano's time, the word chance was used to designate *probability*.

comparing fractions is necessary for probabilistic reasoning in children.

Van Doren (and others) [65] presented some situations like in Cardano's and de Méré's problems to 10^{th} and 12^{th} grade students in an empirical experiment. Before instruction, students had compared events correctly at a qualitative level. Nevertheless, these students erroneously translated their qualitative reasoning using proportional relationships. The illusion of linearity was present and persistent, even after instruction.

2.4 Examples of difficulties in learning probability

We have presented the notion of epistemological obstacle in mathematics and given some examples in probability. Now we present a discussion on some sources of difficulties in learning probability reported in the literature. Some of these difficulties are epistemological obstacles and some are not. It's important to remark that the authors, whose work we discuss below, were not thinking in terms of epistemological obstacles when they've done their research. We don't intend to enter in the whelm of ontogenic or didactical obstacles. The purpose here is to present some research that has been done related to difficulties in learning probability and also some teaching experiences that can illustrate the epistemological obstacles of the previous section. We start with the text of Shaughnessy [57], which is a vast survey of research on the teaching of probability and statistics, what he calls the teaching of stochastics.

With the same concern about mathematical thinking as Hardy and Sierpinska have, Shaughnessy suggests that naive heuristics that are used intuitively by learners impede the conceptual understanding of terms such as sampling, conditional probability and independence (i.e., causal schemes), decision schema (i.e., outcome approach), and the mean. The main themes investigated in his paper are the research on judgmental heuristics and biases that lead to misconceptions and wrong calculations. Learners have difficulties in these areas, however, evidence is contradictory as to whether instruction in stochastics improves performance and decreases misconceptions.

The conclusions emerging from his research are i) probability concepts can and should be introduced in school at an early age, ii) instruction that is designed to confront misconceptions should encourage students to test whether their beliefs coincide with those of others, whether they

are consistent with their own beliefs about other related things, and whether their beliefs come from empirical evidence.

Shaughnessy [57] presents a very broad review of what has been done in terms of research in probability and statistics teaching and learning, more precisely, presenting the misconceptions and difficulties students have in learning stochastics. We will present the ones most relevant to learning probability theory.

The problem of representativeness: people estimate likelihoods for events based on how well an outcome represents some aspect of its population. People believe that a sample (or even a single event) should reflect the distribution of the parent population or should mirror the process by which random events are generated. As an example of the problem of representativeness: in a sequence of boys and girls of a family with 6 children: the sequence BGGBGB is believed to be more likely to happen than BBBBGB or BBBGGG. However, the 3 of them are equally probable.

Representativeness heuristic has also been used to explain the "gambler's fallacy". After a run of heads, tails should be more likely to come up. People try to predict the result that was appearing less often in order to balance the ratio after a small number of trials. Once they have some information about the distribution, even from small sample sizes, they tend to put too much faith in that information. Even very small samples are considered to be representative.

The problem of representativeness is related to the obstacle of proportionality, when people apply a linear reasoning for different sample sizes of an experiment, and also to the obstacle of equiprobability, when people guess the next outcome as the event that will balance the ratio.

The availability problem: the estimation of the likelihood of events are biased by how easy it is to recall such events. If a situation has happened to person A, this person will actually think it's more probable to happen than an objective frequency distribution would tell.

The conjunction fallacy: to rate certain types of conjunctive events more likely to occur than their parent stem events. The reason for saying that $P(A \cap B) > P(A)$ may come from the fact that the event B may have a much higher probability than the event A. Also, people may have a language misunderstanding, when told $P(A \cup B)$, they may understand $P(A|B)$.

Research on conditional probability and independence: One of the most common misconceptions about conditional probability arises when a conditioning event occurs after the event

that it conditions.

As an example: an urn has 2 white and 2 black balls in it. Two balls are drawn without replacement. What's the probability that:

1. The second ball is white, given that the first ball was white? $P(W_2|W_1)$

2. The first ball was white given that the second ball is white? $P(W_1|W_2)$

A common confusion is with the first and the second statements. Many times it's asked $P(W_2|W_1)$ and people usually answer $P(W_1|W_2)$. Other problems show how difficulties in selecting the event to be the conditioning event can lead to misconceptions of conditional probabilities.

Example: There are three cards in a bag. One with both sides green, one with both sides blue and the third one with a blue and a green side. You pull out a card and see that one side is blue. What is the probability that the other side is also blue?

The typical answer, 0.5, assumes a uniform probability, by considering the cards blue-blue and blue-green equiprobable. The correct answer is different, because the 3 sides of the two possible cards are blue, and the blue-blue card has two blue sides, the probability is actually 2/3. This problem is another example of the equiprobability obstacle because we can see a search for a uniform probability where it doesn't apply.

In general, students often confuse $P(A|B)$ with $P(B|A)$. This happens because:

- Students may have difficulty determining which is the conditioning event;
- May confuse condition with causality and investigate $P(A|B)$ when asked for $P(B|A)$;
- May believe that time prevents an event from being the conditioning event like in the white-black balls example;
- May be confused about the semantics of the problem.

It's important to give students examples with time ordered events where the first event is the conditioning one (instead of the second one) to help them overcome the confusion of causality and dependence. Again, as in Hardy [32] and in Schoenfeld [55], students should be given the chance to work on conceptually different tasks instead of only routine ones.

The problem of availability as well as some problems involving conditional probability are not related to the epistemological obstacles that we describe in this theses. Nevertheless, they still count as difficulties of substantial importance in teaching of probability.

Another misconception described in Shaughnessy that can be interpreted in terms of an epistemological obstacle is that people think the real world is filled with deterministic causes and variability is something that doesn't exist to them, because they don't believe in random events. The epistemological obstacle of determinism is discussed in [47].

Although Shaughnessy [57] presents many critiques about the use of naive heuristics instead of mathematical theory, he also says that heuristics can be very useful. The task of the mathematics educators is to point out circumstances which naive heuristics can adversely affect people's decisions and to distinguish these from situations in which such heuristics are helpful.

Many other texts present misconceptions and other difficulties that students face while learning probability at the elementary and secondary level. We only address here two more that may be of interest in the context of this thesis. The first one is Rubel [54], who presents a study on middle and high school students' probabilistic reasoning on coin tasks. The author was interested in the probabilistic constructs of compound events and independence in the context of coin tossing. Ten tasks in probability were assigned to 173 students in grades 5, 7, 9 and 11. They were asked to explain their reasoning. One important result in this paper is that students gave many conflicting answers, reflecting a tension between their beliefs in mathematical thinking. Many of them said that mathematical answers are different from real world answers, which calls attention to the importance of incorporating empirical probability in the classroom, or meaningful situations as suggested by [55] and [23].

The second one is the work of Watson and Moritz [69], that investigates students' beliefs concerning the fairness of a dice. They've interviewed grades 3 to 9 students about their beliefs concerning fairness of dice. An important result to our research interest in this paper is that beliefs based on intuition or classical assumptions concerning equally likely outcomes may be divergent from empirical approaches of gathering data to test such hypotheses. Students presented contradictory answers that indicate a distinction between frequencies and chances; some believed that a few numbers occur more often, but they all have the same chance. Some students have

beliefs in line with the classical approach to probability, based on equally likely cases, which don't always agree with the empirical results of judging probability on long-term relative frequency, as mentioned by Von Plato [67].

To close this section, we present some works that are based on teaching experiments. In a teaching experiment, Shaughnessy [58] would ask students to answer questions about the probability results obtained after performing empirical tasks, such as flipping coins, to confront the empirical results with their intuitions. He found that instruction on formal concepts can improve student's intuitive ideas of probability and also reduces reliance upon heuristics. Not all the students overcame the misconceptions because conceptual change takes time and a great deal of effort to happen.

Freudenthal [23] interprets probability as an application of mathematics with very low demand of technically formalized mathematics and as an accessible field to demonstrate what mathematics really means. According to him, probability is taught as an abstract system disconnected from reality or as patterns of computations to be filled out with data. He regrets a theoretical teaching approach and prefers a non-axiomatic teaching style. To Freudenthal, if probability is taught through its applications, "*axiomatics is not much more than a meaningless ornament*" (p. 613). We don't share this opinion, because as it will be shown in chapter 5, a formal axiomatic approach could solve probability problems free of ambiguities. At the same time, caution must be taken, because, as mentioned by Schoenfeld [55], tasks must be meaningful and stimulate mathematical thinking. Hence we advocate that an axiomatic teaching approach sets the students with the tools to face problems free of ambiguities; while we agree that instruction and problems have to meaningful and related to real questions, we - as does Schoenfeld - advocate that have to be related to the problems that made science progress.

It's important to say that in general, formal instruction is not enough to overcome misconceptions. Students need to confront the misuses and abuses of statistics and to *experience* how misconceptions of probability can lead to erroneous decisions. In other words, it's important that students confront their epistemological obstacles for an act of understanding to take place. An instructor *showing* misconceptions and refuting them in front of the students in a lecture does not necessarily lead to students being more critical and more relying on theoretical reasoning than on

guessing and intuition. This was one of the conclusions from Miszaniec [47]. More subtle teaching situations have to be devised, as suggested in Schoenfeld [55].

2.5 Closing remarks

We have presented the discussion of the concept of epistemological obstacles in the works of Brosseau [16] and Sierpinska [60], [61], [62]. Hardy [32] and Schoenfeld [55] discuss the role of non-routine tasks in setting students in a path towards mathematical behaviour and reasoning when solving problems. All of those authors, advocate that epistemological obstacles, instead of being avoided, should be confronted in order to advance in the acts of learning.

Applying the concept of epistemological obstacle to probability, we've seen, as examples, the obstacle of determinism, the obstacle of equiprobability and the illusion of linearity. Those three obstacles were key in the evolution of probability and had to be overcame for the theory of probability to advance. In this thesis we focus in the obstacles of equiprobability and proportionality. The obstacle of equiprobability has been investigated by Lecoutre [45], Gauvrit, Morsanyi and others [25], [48], using tasks involving a non-uniform random variable obtained from the combination of two or more uniform random variables or in tasks involving different sample sizes. The obstacle of proportionality has been commonly studied in situations involving binomial experiments, by Van Dooren (and others) [65], [20] and also by Miszaniec [47]. There is a gap in the study of these obstacles when we think of the modern definition of probability, specially when using infinite spaces.

We've presented some research that has been done regarding students' difficulties in learning probability. Many of those difficulties from the previous section are examples of epistemological obstacles. Hardy [32], Schoenfeld [55], Shaughnessy [58] and Freudenthal [23] discuss the importance of non-routine tasks leading to unexpected results for the learning of mathematics. Nevertheless, Freudenthal has a dissonant view from the others when he qualifies set and measure theoretical probability as an old-fashined teaching approach and the axiomatization a meaningless ornament. Shaughessy [58] suggests that students should do practical experiments to confront their beliefs prior to instruction, in a sense that is close to the experiences reported by Sierpinska

[60]. Schoenfeld [55] discusses general mathematical learning, and he says that the tasks must have meaning by being connected to the problems that made science progress so the students would engage in mathematical thinking, just like Hardy [32] suggests.

In this literature review, we observed a lack of studies regarding the (axiomatic) definition of probability at the post-secondary level and this thesis aims to contribute to filling in this gap. Many studies have been done concerning elementary or high school students and many of those are dedicated to conditional probability, randomness, representativiness but there is little or no research on students' understanding of the definition of probability at the post-secondary level.

Chapter 3

A pilot study into graduate students' misconceptions in probability

3.1 Introduction

How familiar are students with the modern definition of probability? Are they aware of the axiomatic definition? Do they fall into the obstacles of equiprobability or proportionality when they conceptualize probability? What approach do they use to handle infinite sample spaces? Do they think of a σ-algebra or is their reasoning still based on favorable over possible outcomes? This chapter presents a pilot study into students' awareness of the modern approach to probability, based on the axioms of Kolmogorov.

When dealing with infinite spaces, a classical approach to probability, based on the ratio between favorable and possible cases, is ineffective but still very commonly used. This is a source of the obstacles of proportionality and equiprobability, as seen in chapter 2. This pilot study was originally conceived as a first exploratory study with the original purpose of verifying the hypothesis that the classical approach is still present in students' minds when they think of probability, even at the graduate level. The main result found is that only one student who is finishing his doctoral research in probability used the modern approach, while all the other graduate students interviewed still recall the classical approach. An unexpected result is that they still fall into the proportionality and equiprobability obstacles. This result motivated us to investigate the treatment

that the books give to the definition of probability on chapter 6.

We are aware of the limitations associated to a preliminary study conducted with a very small sample of a specialized population – graduate students in mathematics and statistics. In future research, this study could be extended to a larger sample of students from different universities and also from different areas that are highly connected to probability, such as engineering or computer science. However, this thesis focuses on the ideas enhanced by didactic books vis-à-vis the birth of modern probability. Considering that the adopted book is an important source of theoretical knowledge and its exercises are a guide to understand the most important ideas developed in the text, this pilot test was used to justify the analysis of the textbooks that we do in this thesis.

Following this introduction, the second section gives an overview of the study, the students we've interviewed and the questions we've asked them. The third section is a brief description of the method of data analysis. Section four presents the results that we've found with the students. Section five presents a discussion and section seven some final remarks.

3.2 Overview of the pilot study

The research instrument is an interview devised with the support of two probability books. The first one is from Shiryaev [59], who studied directly under the supervision of Kolmogorov. We used this book to elaborate upon the definition of probability presented to the students. The second textbook is Grinstead and Snell [28], the only undergraduate level book where we found exercises that makes one think about the definition of probability in infinite spaces. This difficulty in finding textbooks with these type of exercises made us curious about the treatment given to probability in other textbooks.

For the interview, we recruited four graduate students from the department of mathematics and statistics of Concordia University. The purpuse was to have subjects with a good probability background, who have been in touch with probability in the last six months, so they would have the ideas and concepts "fresh" in their minds. Two students were from the PhD program and the other two were from the MSc program. For the purposes of identification, we labelled the two PhD students as PhD 1 and PhD 2 and the two master students as MSc 3 and MSc 4. PhD 1 had just

passed the comprehensive exam a few weeks prior to the interview, with probability as was one of the covered topics. PhD 2 was to complete her/his thesis within a year, and is doing research in probability. MSc 3 and MSc 4 are both first year students who are presently taking a graduate course in probability, with their interviews taking place just a few days before their final exam. We used PhD 1's interview as a preliminary trial, to see if we would need to modify the questions. As the interview was validated, we applied it to the other participants.

Each interview was conducted individually with the participants. We sat next to the student, we gave them sheets with the questions and then we explained that every answer should be justified in the best way they could. If they were not able to give a formal mathematical justification, they could explain their thoughts and intuition using words. By being beside the student, we could make sure that the participants had a good understanding of the question and, in case they could not write their answer, they were able to explain their thoughts to me verbally. When this was the case, we wrote down the answer and showed it to the student to verify that this was a good representation of what they thought.

The main goal of the interview was to see if the students were familiar with and used the modern definition of probability, based on Kolmogorov's axioms and as a result, we've identified the persistence of the epistemological obstacles of the proportionality and equiprobability. We will present the questions and results that bring insight into student's conceptualization of probability and the epistemological obstacles we've identified in their answers.

In part A, there are four questions about the relationship between probability and sets of measure zero. This is particularly important because it is related to Poincaré's intuition that probability 0 doesn't necessarily mean an impossible event and probability 1 doesn't indicate a certain event. This intuition contradicts the idea of classic probability based on Cournot's principle: "*An event with very small probability is morally impossible: it will not happen. Equivalently, an event with very high probability is morally certain: it will happen*" [56] (p. 72). This principle was first formulated by Bernoulli [7] and developed by Antoine Augustin Cournot [18]. This epistemological obstacle was overcame by Kolmogorov's foundation of modern probability, where probability 0 events are not seen as impossible anymore.

In part B, we asked the students to define probability and then compare their definitions to

a formal definition based on Kolmogorov's axioms. We then introduced questions inspired on Grinstead and Snell [28]. The questions were on countable additivity, which is another important property brought by modern probability theory. We also questioned if it is possible to define probability in the classical way in countable infinite spaces. This is another limitation of classic probability that modern probability was able to overcome.

Even though through all the parts of the interview we're interested in probing participants' understanding of measure theoretical concepts such as probability measure, sets of measure zero and countable additivity, a background in measure theory is not necessary to answer any of the questions. The interest lies in figuring out if the student uses the concept of probability according to Kolmogorov's axioms, rather than the classic approach. The way we chose to reveal students' perception of the modern definition is to expose them to situations where they have to handle infinite sample spaces. The unexpected result is that the epistemological obstacles of equiprobability and proportionality were found in graduate students. The interview is described in detail below, with the interview text in italics and the comments that were not shown to the students presented in regular characters.

3.2.1 Part A – Questions on some properties of probability

In the first question, we want to see if students will use proportional reason in a situation where it does not apply, that is the illusion of linearity.

Question A1: *Suppose that one person is testing two cars on a road. The cars are of the same model, year, and type of motor. The weather conditions are the same as well as the car's driver. The trip starts from point A and the distance the cars can travel on that road is a function of the amount of fuel they have. The first car has the fuel tank filled up to 1/4 and the probability of reaching point B on that road is 0.6. The second car has the fuel tank filled up to 1/2. So the probability that the second car reaches point B on that road is:*

a) More than twice as much as the 1st car.

b) Twice as much as the as the 1st car.

c) Less than twice as much as the 1st car. [right answer]

Since the probability for the first run is 0.6, the probability can't be a linear function of the

amount of fuel, otherwise twice more fuel would imply a probability greater than 1. The goal is to see whether the student falls into the illusion of linearity.

Questions A2 to A5 are testing whether students are aware of the role played by sets of measure zero in probability. More specifically, we ask students if they are aware that:

(1) If A is an impossible event, then $P(A) = 0$, but the converse may fail. (This statement is tested in question A4 and the converse is tested in question A2);

(2) If A is a certain event, then $P(A) = 1$, but the converse may fail. (This statement is tested in question A5 and the converse is tested in question A3).

For questions A2 to A5, let A be an event and $P(A)$ be the probability that the event A happens. Read the following statements and write whether you agree or not with them. Justify each answer based on the probability theory that you learned in your academic life.

Questions A2-A5

A2) If I have $P(A) = 0$, then it is impossible that the event A will happen. [False]

A3) Even if I have $P(A) = 1$, the event A may still not happen.[True]

A4) If I know for sure that the event A will not happen, then I can say that $P(A) = 0$. [True]

A5) If I know for sure that the event A will happen,then I can say that $P(A) = 1$. [True]

These questions are related to Poincaré's intuition that with an infinity of possible results, probability 0 doesn't necessarily require the event to be impossible as well as probability one doesn't necessarily mean the event is certain [51]. It is also related to the discussion on proportional reasoning, that is often and mistakenly used when looking at the non-linear distance of experiments whose probability are close to 0 or to 1.

3.2.2 Part B – Questions on countable infinite sample spaces

The questions in part B are all connected. Starting from the definition of probability, if students consider the sample space as a set of equiprobable events and probability as a proportion of favourable over possible cases, as described in the previous chapter,they have a classical conceptualization of probability and have not overcome the epistemological obstacles of equiprobability

and proportionality yet. The goal of the questions in this part B is to make students find a contradiction if their conceptualization of probability is the classical one, otherwise, no contradiction should be found if they use a modern approach.

Part B starts with two warm up questions. The interest lies in the students' conceptualization of probability. They compare their definition with a formal one and answer a simple question that can be resolved with the classic probability approach. We wanted to see if the student would use a modern or a classic approach even after thinking of the definition of probability and validating her/his definition with a formal and axiomatic one.

Warm up question B.1: *Do you remember the definition of probability? State it as formally as you can and then check it with the definition on page 4.*

Warm up question B.2: *Think of a die. What is the probability that you will get the number 4 in a roll of a die? What is the probability that you will get an odd number in a roll of a die? How did you find these results?*

If the student easily recalls a modern conceptualization of probability and is aware of the difference with the classical approach, the explanation of the probability found in the trivial die experiment should be explained using the axioms, instead of a rate of favourable over possible cases.

In questions B1 and B2, we expected intuitive answers. The goal was to make students think about assigning probability in countably infinite sets using the classical approach in question B1 and considering equiprobable events in question B2.

Questions B1-B2

B1) Think of a countably infinite set. Can we assign probability to each element of this set by the ratio between the number of favorable cases and the number of all possible cases?

B2) Is it possible to define a probability function uniformly distributed on the natural numbers, $\mathbb{N}$?

We would expect a negative answer in both questions from a student who is familiar with the modern approach. Question B1 is useful to identify the presence of a proportional reasoning and question B2 is useful to identify the presence of equiprobability.

Question B3 remains in the intuitive realm of countable infinite sets, like questions B1 and B2, but now we start giving the first step to build (or not) the contradiction as we've explained at the

beginning of this part B.

Question B3: *What, intuitively, is the probability that a "randomly chosen" natural number is a multiple of 3?*

We were expecting the intuitive answer of 1/3 from all of them, but the justification is what makes an important difference. Modern probability enable us with a σ-algebra of sets that allows us to assign probability to certain subsets of our probability space without passing through the ratio of favourable over possible cases.

Question B4 is not really a question looking for an answer from the student, rather, the goal of this question is to guide the student to a possible way of assigning "probabilities" in a classical way to a countably infinite set.

Question B4: *Let $P(3N)$ be the probability that a natural number, randomly chosen in {1, 2, ..., N}, is a multiple of 3. Can you see that $\lim_{N\to\infty} P(3N) = 1/3$? Let's call this limit $P3$. This formalizes the intuition in question B3, and gives us a way to assign "probabilities" to certain events that are infinite subsets of natural numbers.*

In question B5, we expect students to find a contradiction with the "probability" defined in question B4 and that countable additivity fails.

Question B5: *If A is any set of natural numbers, let $A(N)$ be the number of elements of A which are less than or equal to N. Then denote the "probability" of A as $P(A) = \lim_{N\to\infty} A(N)/N$ provided this limit exists. What is the probability of A, if A is finite? And if A is infinite? Do you see any contradiction with $\lim_{N\to\infty} P(3N) = 1/3$ from question B4?*

This question is really important because the expected answer is: i) $\lim_{N\to\infty} A(N)/N = 0$ when A is finite, and ii) there is a bijection between any infinite subset $A \subset \mathbb{N}$ and $\mathbb{N}$ itself, so their cardinality is the same and the answer is 1 when A is infinite. This creates a contradiction with question B4, where $\lim_{N\to\infty} P(3N) = 1/3$. This contradiction is interesting because it shows the students how the epistemological obstacles of proportionality and equiprobability from the classical approach lead to contradictions that were resolved by Kolmogorov's axioms.

Questions B6 and B7 are linked to question B5. In question B6 we want to see if the student is aware of countable additivity of probability and in question B7 we want to put in evidence that in infinite sets, equiprobability is not a valid assuption about the events.

Questions B6-B7

B6) Let $\Omega = \mathbb{N}$. *Is it true that if* $\mathbb{N} = \cup_{i=1}^{\infty} n_i$ *, then* $P(\mathbb{N}) = \sum_{i=1}^{\infty} P(n_i)$*?*

B7) We know that: if $n_1, n_2, ...,$ *is a countable infinite sequence of disjoint subsets of* $\mathcal{F}$*, then* $P(\cup_{i=1}^{\infty} n_i) = \sum_{i=1}^{\infty} P(n_i)$*. Is it compatible with the question B6, in the sense that* $\mathbb{N} = \cup_{i=1} n_i$*, then* $P(\mathbb{N}) = \sum_{i=1}^{\infty} P(n_i)$*?*

We would expect the students to agree with both statements if the use a modern approach. That not being the case, the student would not be sure of questions B6 and B7 after the contradiction presented in question B5.

Question B8 is a "meta-cognitive" question, in the sense that makes the student think about what he has developed in these questions and evaluate if his perception of probability has changed or not.

Question B8: *Go back to question B1. Have you changed your mind? Justify.*

In this question we expect students to change their minds if they would think that it is possible to assign a probability to each element of a countable infinite set through the classical approach.

3.3 Methods of data analysis

To analyze the answers, we compare the participants' answer with the right answer and also with one another's answer. For each question we set the answers in a table, followed by the students' reasoning and the analysis of their answer based on four different possibilities: i) the student used a correct argument to answer, ii) the student had some misconceptions, which reflect a wrong or inaccurate idea about a mathematical concept, iii) the student use a *false rule*, which is a procedure or technique that the student applies that is not true according to the theory, and iv) the student encounters a *difficulty*, which is the incapacity to find a solution for a question or to organize and express her/his thoughts.

3.4 Results

We analyze students' answers to the questions in the tables bellow. The answer to each question is presented with the student's answer and reasoning, and our analysis.

Table 3.1: Questions A1

Question A1 Suppose that one person is testing two cars on a road. The cars are of the same model, year, and type of motor. The weather conditions are the same as well as the car's driver. The trip starts from point A and the distance the cars can travel on that road is a function of the amount of fuel they have. The first car has the fuel tank filled up to 1/4 and the probability of reaching point B on that road is 0.6. The second car has the fuel tank filled up to 1/2. So the probability that the second car reaches point B on that road is: a) More than twice as much as the 1st car. b) Twice as much as the as the 1st car. c) Less than twice as much as the 1st car.			
Student	**Answer**	**Reasoning**	**Analysis**
PhD1	c	"The probability can't go beyond 1"	Right answer, since option "c" was the only one that would be in the interval $[0, 1]$.
PhD2	a	"The problem didn't mention if the function was linear or not". The probability of getting to "B" is 0.6, so he took the complement: $1 - 0.6 = 0.4$. Student stopped here.	Misconception that the probability could be outside the interval [0, 1]. I asked him why he chose "a" and he said that as he didn't know the function, he just took a guess.
MSc1	b	"Twice the fuel should be twice the distance"	False rule: student erroneously applied linear reasoning.
MSc2	c	"The probability can't go beyond 1"	Right answer, since option "c" was the only one that would be in the interval [0, 1].

Table 3.2: Questions A2 to A5

Question A2 to A5 A2) If I have $P(A) = 0$, then it is impossible that the event A will happen. A3) Even if I have $P(A) = 1$, the event A may still not happen. A4) If I know for sure that the event A will not happen, then I can say that $P(A) = 0$. A5) If I know for sure that the event A will happen, I can say that $P(A) = 1$.			
Student	**Answer**	**Reasoning**	**Analysis**
PhD 1 and 2	Both students didn't agree on the 1^{st} and agreed with the other sentences.	Argument: A is an impossible event $\Rightarrow P(A) = 0$, but the converse may fail. A is a certain event $\Rightarrow P(A) = 1$, but the converse may fail.	Right answer and they also identified that questions 4 and 5 complement each other, just like questions 6 and 7.
MSc1	disagreed with 5	"$P(A) = 0 \Rightarrow A$ is an impossible event". Student has written the statement above but when I asked him to explain what he was thinking about, he said, I don't know how to explain it to you. I only know that if the probability is zero, the event can't happen and if the probability is one it must happen. Then I asked him if the converse of these statements were true or false and he said the converse is also true.	False rules: $P(A) = 0 \Rightarrow A$ is an impossible event. $P(A) = 1 \Rightarrow A$ is a certain event.
MSc2	disagreed with 4 and 5	Student couldn't write a justification, but he said that "A is an impossible event $\Rightarrow P(A) = 0$, but the converse may fail. $P(A) = 1 \Rightarrow A$ is a certain event"	False rule: $P(A) = 1 \Rightarrow A$ is a certain event.

Table 3.3: Warm up question B.1

Warm up question B.1 State the definition of probability as formally as you can and then check it with the definition on page 4.		
Student	**Answer**	**Analysis**
PhD1	"a function that gives a degree of uncertainty about an event and takes values between 0 and 1".	Difficulty: student gave a Bayesian interpretation of probability, but he didn't define it axiomatically or state its properties.
PhD2	"Let $\mathcal{F} :\rightarrow R$ and $P :\rightarrow [0,1]$, $P(\emptyset) = 0, P(\Omega) = 1$. P is a sigma additive measure".	Misconception: student defined the sigma algebra $\mathcal{F}$ as a function and didn't specify its domain. He confused it with a random variable, which is in fact a function. Student defined a probability function P, but he didn't say that the domain should be $\mathcal{F}$. Student mentioned sigma additivity, which can be seen as countable additivity, but he didn't mention the monotonicity.
MSc1	"Probability is a measure which gives (define) the likelihood of an event to happen".	Difficulty: student could not remember the definition and didn't mention any property of the probability function. Misconception (epistemological obstacle): circular definition, like in the classical approach using likelihood to define probability.
MSc2	"In a σ-field and a probability space Ω, we can find a real-valued function from 0 to 1 that maps to the outcome of the event. The value is called probability".	Misconception: Student didn't know that to become a probability space, a measurable space must have a probability function defined. The mapping should be from the event to $[0,1]$ and not the opposite. Student didn't mention any of the properties of probability.

Table 3.4: Warm up question B.2

Warm up question B.2 Think of a die. What is the probability that you will get the number 4 in a roll of a die? What is the probability that you will get an odd number in a roll of a die? How did you find these results?			
Student	**Answer**	**Reasoning**	**Analysis**
PhD2	1/6 and 1/2	"$1 = \sum_{i=1}^{6} P(A_i) = 6P(A_1) \Rightarrow P(A_i) = 1/6$, considering the symmetry of the die. Analogous way to the 1/2."	It was the only student that didn't mention the classical approach.
All the other students	1/6 and 1/2	Ratio between favorable and possible cases.	All of them gave correct values of the probabilities, but using the classical approach.

Table 3.5: Question B1

Question B1 Think of a countably infinite set. Can we assign probability to each element of this set by the ratio between the number of favorable cases and the number of all possible cases?			
Student	**Answer**	**Reasoning**	**Analysis**
PhD1, PhD2 and MSc2	No	The three participants presented that, in this case, for the sum to converge, each element should have probability 0, so in this case the sum wouldn't be 1.	The three presented the right reasoning. They were thinking about countable additivity.
MSc1	yes	This student couldn't answer the question and skipped it. But he came back after resolving question 5. He said it would be: "Yes because the number of elements of the set would be infinite".	Difficulty: student wasn't able to express his thoughts and he got more confused when he came back to the question after doing question 5.

Table 3.6: Question B2

Question B2 Is it possible to define a probability function uniformly distributed on the natural numbers, $\mathbb{N}$?			
Student	**Answer**	**Reasoning**	**Analysis**
All the students	No	For the sum to converge, each element should have probability 0, so in this case the sum wouldn't be 1.	The four presented the right reasoning. They were thinking about countable additivity.

Table 3.7: Question B3

Question B3 What, intuitively, is the probability that a "randomly chosen" natural number is a multiple of 3?			
Student	**Answer**	**Reasoning**	**Analysis**
PhD2	1/3	He made a partition of $\mathbb{N}$ in equivalence classes of the natural numbers mod(3). Then he set probability 1/3 for each class, so the class of 0 (which are the multiples of 3) would have probability 1/3.	This answer is more sophisticated than the classical approach. He split the natural numbers into 3 equivalence classes that have the same cardinality, so they are symmetric with regards to the counting measure.
All the other students	1/3	All the others thought about the ratio between favorable and possible cases.	False rule: they all applied the classical approach to an infinite set.

Table 3.8: Question B4

Question B4 Let $P(3N)$ be the probability that a natural number, randomly chosen in $\{1, 2, ..., N\}$, is a multiple of 3. Can you see that $\lim_{N\to\infty} P(3N) = 1/3$? Let's call this limit $P3$. This formalizes the intuition in question B3, and gives us a way to assign "probabilities" to certain events that are infinite subsets of natural numbers.			
Student	**Answer**	**Reasoning**	**Analysis**
PhD2	1/3	He made a partition of $\mathbb{N}$ in equivalence classes of the natural numbers mod(3). Then he set probability 1/3 for each class, so the class of 0 (which are the multiples of 3) would have probability 1/3.	This answer is more sophisticated than the classical approach. He split the natural numbers into 3 equivalence classes that have the same cardinality, so they are symmetric with regards to the counting measure.
All the other students	1/3	All the others thought about the ratio between favorable and possible cases.	False rule: they all applied the classical approach to an infinite set.

Table 3.9: Question B5

Question B5 If A is any set of natural numbers, let $A(N)$ be the number of elements of A which are less than or equal to N. Then denote the "probability" of A as $P(A) = \lim_{N\to\infty} A(N)/N$ provided this limit exists. What is the probability of A, if A is finite? And if A is infinite? Do you see any contradiction with $\lim_{N\to\infty} P(3N) = 1/3$ from question B4?			
Student	**Answer**	**Reasoning**	**Analysis**
PhD1	A finite: 0; A infinite: 1; No contradiction.	"$A(N)/N \to 0$ as $N \to \infty$. $N/N \to 1$ as $N \to \infty$. No contradiction."	The first two answers are right, however, there is a difficulty when he didn't see the contradiction. He should see that countable additivity wouldn't work for the subsets of $\mathbb{N}$.
PhD2	A finite: 0; A infinite: no answer. No contradiction.	He saw clearly that when A is finite, $P(A)$ goes to zero. For the infinite case, he started an argument using the power set of A. I tried to make it simpler and said, instead of a generic set, think of $\mathbb{N}$, and he thought that the power set of $\mathbb{N}$ was a subset of it.	For the finite case he gave the right answer. For the infinite case, there was a misconception that the power set of $\mathbb{N}$ was a subset of $\mathbb{N}$. He saw no contradiction, because he never thought of the probability with the classical approach in the previous cases.
MSc1	A finite: 0; Indetermination. No contradiction.	He saw clearly that when A is finite, $P(A)$ goes to zero. He found that the limit would be an indeterminate of the form ∞/∞, which can be anything. No contradiction because of the indetermination found.	For the finite case the answer is right. For the infinite case: difficulty – instead of taking the cardinality of the sets to establish N/N and find 1 as a result, he put ∞/∞.
MSc2	A finite: 0; Indetermination. No contradiction.	He saw clearly that when A is finite, $P(A)$ goes to zero. He found that the limit would be an indeterminate of the form ∞/∞, which can be anything. "I don't have enough arguments to see any contradiction".	For the finite case the answer is right. For the infinite case: difficulty – instead of taking the cardinality of the sets to establish N/N and find 1 as a result, he put ∞/∞. As the student didn't see that the limit was 1 in the infinite case, he couldn't see the contradiction.

Table 3.10: Question B6

Question B6: Let $\Omega = \mathbb{N}$. Is it true that if $\mathbb{N} = \cup_{i=1}^{\infty} n_i$, then $P(\mathbb{N}) = \sum_{i=1}^{\infty} P(n_i)$?			
Student	**Answer**	**Reasoning**	**Analysis**
PhD1	True	"Just think of countable additivity".	The answer is not wrong, but there was a difficulty in stating the probabilities for each n_i, that was not done.
PhD2	True	He said that the idea is to build the probability for each n_i in the same way he built for the multiples of 3 (with equivalence classes) and then, the statement is true.	Difficulty – the student didn't get to actually build the probability for n_i using equivalence classes. He couldn't define those equivalence classes.
MSc1	False	"It's impossible to do it because if each n_i have the same probability, the sum diverges".	The answer is correct, but incomplete, because he could define the probability like a geometric series: $P(n_i) = 2^{-n_i}$
MSc2	True	"I don't know how to justify it".	

Table 3.11: Question B7

Question B7 We know that: if $n_1, n_2, ...,$ is a countable infinite sequence of disjoint subsets of $\mathcal{F}$, then $P(\cup_{i=1}^{\infty} n_i) = \sum_{i=1}^{\infty} P(n_i)$. Is it compatible with the question B6, in the sense that $\mathbb{N} = \cup_{i=1} n_i$, then $P(\mathbb{N}) = \sum_{i=1}^{\infty} P(n_i)$?			
Student	**answers**	**Reasoning**	**Analysis**
PhD2	True	He used the same argument as in question B6.	Difficulty – the student didn't build the probability for n_i using equivalence classes. He couldn't define those equivalence classes.
MSc1	False	"For this to be true, we would need many n_i with probability 0".	Difficulty: a positive, but geometrically decreasing probability would be true as well, however it is true that for i large enough, $P(n_i)$ would need to be smaller than ε.
MSc2	True	"I don't know how to justify it".	

Table 3.12: Question B8

Question B8: Go back to question B1. Have you changed your mind? Justify.			
Student	**Answer**	**Reasoning**	**Analysis**
PhD1	Yes	"I changed my mind because according to questions 3, 4 and 5 we can build this probability".	Difficulty: the student didn't understand that question five gives a contradiction with the way of assigning probability in question 4 for infinite sets.
PhD2	No		He didn't see a contradiction because from the beginning he was already thinking of probability as a measure, instead of a classical approach.
MSc1	Yes	"With the way we defined probability in question 4 we can assign probability like in question 1."	Difficulty: the student didn't understand that question five gives a contradiction with the way of assigning probability in question 4 for infinite sets.
MSc1	No	No justification.	He couldn't justify because he didn't see the contradiction. He thought he needed more information.

3.5 Discussion

In part A, it's surprising that the student who performed best with the definition of probability, PhD2, got lost in question A1, that probability can't be bigger than 1. Not only PhD2, but MSc1 also fell into the proportionality obstacle, when they applied a linear reasoning in a situation where it is not possible.

Regarding questions A2 to A5, both PhD students were aware of sets of measure zero and that they can represent events which are not necessarily impossible. Both MSc students were not sure about these results and made mistakes about probability 0 and impossibility or probability 1 and certainty of events.

In part B, the first result that calls attention is that PhD2 is the student with most familiarity with probability. He was the only one who, since the beginning, used the modern approach instead of the classical one. This explains why he could see clearly that it's not possible to define a probability with the classical approach in an infinite set. He made some mistakes through the interview but this can be attributed to distraction or lack of concentration.

MSc1 was the student that has shown the most contradictory answers. In Part B, he had difficulty answering question 1 and chose to skip it and move ahead, but then correctly answered question 2, which focuses on a very similar idea.

PhD1, MSc1 and 2 gave answers at the same level of comprehension. They took a classical approach in question 3 and didn't see the contradiction of this approach to countably infinite sets.

Also, PhD1 and MSc2 started with the idea that it is not possible to give a uniform probability to the natural numbers, but they changed their minds when they didn't identify the contradiction between questions 4 and 5 and countable additivity. The element that was clear to them is that the sum of the probabilities must be no greater than 1.

All the four students fell into the illusion of linearity and/or the obstacle of equiprobability, which, despite the limitations of this pilot study, indicates a future research direction. The persistance of those epistemological obstacles also made us curious about the approach that textbooks advance most. After the interview, We reviewed with them the questions comparing their answers to the expected ones as a form of feedback. PhD2 said that he had a lot of fun during this interview because the subject was very interesting. The other three students were all glad that they had participated in the interview, and all four said that they had learned something about probability as a

result of the questions.

The results found in this study is that students think of probability using the classical approach, however, as demonstrated by PhD2, this view can change as you mature in the subject. Another surprising result to us is that the epistemological obstacles of the illusion of linearity and equiprobability are persistent among these graduate students. Nevertheless, caution must be taken, because this is just a pilot study with only four students. These results must be seen as a first insight into the questions discussed here, and using them to make inferences about a wider population would be another epistemological obstacle in probability, called law of small numbers [57]! This happens when the results of a small and non-representative sample are extrapolated to a big population.

3.6 Final remarks

This pilot study was conceived to explore graduate student's conceptualization of probability. Regarding the approach they use in probability, it was shown that, except the student finishing her/his PhD research in probability, all the others graduate students are more inclined to a classical than to a modern one. Their answers pointed to confusion when asked to deal with infinite sets. In particular, contradictions were found on whether it is possible or not to use the classical approach to assign probability to infinite sets. This puts in evidence the persistence of the epistemological obstacles of equiprobability and proportionality, that are associated to a classical reasoning in probability.

This experiment can be improved in some ways. A bigger and more diversified sample with students from other domains that use a lot of probability can always bring better and safer insights. Also, if the student is certain that in question 5 of part B if the probability is 1 when A is an infinite set, and if we relate it more clearly with countable additivity, the quality of the answers for analysis may be improved, because these items are essential to find a contradiction with the classical approach of probability in infinite spaces.

Chapter 4

Classical Probability: The Origins, Its Limitations and the Path to the Modern Approach

4.1 Introduction

In this chapter, we want to discuss why probability became attached to measure theory at the beginning of the 20^{th} century. More specifically, we are interested in knowing why probability *needed* measure theory as its basis to be considered an autonomous branch of mathematics. While probability has been present in various branches mathematics for many centuries, it was not until the development of measure theory in the late 19^{th} and early 20^{th} century that probability could be developed in full mathematical rigour. Following the work of such mathematicians as Borel, Lebesgue and Fréchet, a strong relationship between probability and measure theory became apparent.

If probability existed for centuries in mathematics before the development of measure theory, why did the former *needed* the latter to constitute its basis? Which mathematical problems of the time relied on the understanding of probability as a measure? What was the motivation for this theoretical view change in probability, which had driven an association of probability and measure at the very early stage of development of measure theory?

Science doesn't progress in a linearly path. From one advance to a new discover, there is a

myriad of distinct paths by which to continue, many of them leading down wrong turns, labyrinths of blind alleyways, or dead ends. This road, full of sinuous curves makes the progress of science slow. For example, Borel was studying convergence of series in complex analysis when he first forayed into measure theory. Rather than proceeding with a purely chronological exposition, we will explore the main ideas, even the blind alleyways, that led to the axiomatization of probability based on measure theory.

Prior to Kolomogorov's axiomization of probability in 1933, classical probability was considered a branch of applied mathematics. It provided formulas for error terms, economic activities, statistical physics and solutions to problems in games of chance. This non-mathematical context was related to combinatorics and differential equations among others. Despite the advances in classical probability, not much attention was given to the mathematical basis of that probabilistic context, and the subject was not yet considered an autonomous branch of mathematics. It was connected with a finite number of alternative results of a trial that are considered equiprobable but *"... even the real world does not possess the absolute symmetries of the classical theory's equipossible cases"* [67] (p. 6). The concepts and methods were specific to applications, and their contributions to larger questions of science and philosophy were limited. Regarding the mathematical point of view, there was a need for the definition and foundation of probability using a general and abstract approach. Before this formalization could be achieved, the development of measure theory was necessary, so probability could use it as the ground for its modern foundations and to become an autonomous mathematical discipline as we will see in the following sections.

In a broader perspective, the shift from classical to modern probability appears as part of a greater movement, the very change from classical to modern science itself. Von Plato [67] saw that it would be necessary to find a scenario requiring the development of the concepts of chance and statistical law, for probability to become an autonomous branch of mathematics. Although mathematicians had begun looking for a formal and abstract definition of probability before the turn of the century, it was not until the quantum mechanical revolution between 1925 and 1927, that the abstract study of probability became necessary for further scientific advancement. Quantum mechanics viewed the elementary processes in nature as non-deterministic, with probability playing an essential role in describing those processes. In its relation to physics, probability had many technical developments motivated by statistical physics, however the foundation for the development of modern probability found a ground in quantum mechanics, studied by Hilbert and

Kolmogorov himself.

In this chapter, we will discuss the development of the set of axioms for probability based on measure theory, that is, a very deep change in the basis of probability that took it from a set of tools to solve problems from physics, gambling, economics, and other human activities to an autonomous branch of mathematics. This period in the history of probability is analyzed by Shafer and Vovk [56]. They advocate that Kolmogorov's work in establishing a set of axioms was a product of its time, in the sense that the emergence of these ideas does not stem exclusively from Kolmogorov's originality, but is due to the presence of the work of many of his predecessors. We will use the historical approach used by Shafer and Vovk as a guideline, however our study will explore a narrower account of history, focusing only on those fewer ideas that we consider to be key concepts in the establishment of the axioms and providing for these results a more detailed mathematical exposition. We will occasionally detail succinct proofs from their original sources, providing insight to make them more accessible and closer to today's language.

Another important source in this subject is the book of Von Plato [67]. He presents many problems that motivated the development of probability as well as some philosophical questions. His approach differs from this chapter in that it is more concerned with the development of the philosophy and the concepts of probability in connection to statistical and quantum physics. We have chosen to focus instead on the mathematical features of the development of modern probability. The philosophy and different interpretations of probability, although very interesting subjects, go beyond the scope of our work and can be themes for another thesis.

In the next section, we will concern ourselves with probability before to the 20^{th} century. We will discuss the origins of probability, the definition of classic probability of Bernoulli and De Moivre that remained essentially stable until the birth of measure theory, and Bayes' contribution for the cases involving the dependence of events. The third section presents the development of measure theory, with focus on the results that were important to the development of modern probability. In the last section we will discuss the natural association of probability to measure theory, present since its inception with the work of Borel and Lebesgue. We will explain the association of both disciplines, the need to develop a general and abstract set of axioms for probability and the first attempts at an axiomatization. We will also discuss Borel's denumerable probability, more specifically the use of countable additivity and the strong law of large numbers, two essential results to the foundation of the axioms.

4.2 Probability before 1900

In this section, we start by presenting the origins of probability and its establishment to the status of a science. We present Bernoulli's book, *Ars Conjectandi*, focusing on two features that are central to this thesis: i) the classical definition of probability and ii) Bernoulli's law of large numbers, which was the first convergence theorem in probability that was presented and proved with complete analytic rigour. After Bernoulli's work we present the work of De Moivre, *The Doctrine of Chances*, and conclude with Bayes' contribution to conditional probability with the theorem that carries his name.

4.2.1 The origins of probability

It is widely accepted that the birth of and early developments in probability theory arose from gambling. Even without denying the importance of gambling to the development of probability techniques, Maistrov [46] asserts that probability theory could emerge only after the problems connected with probabilistic estimation from several fields of human activity became more pressing. The turn of the century brought about a period of the collapse of feudal relations, proletarization of peasants and the rise of the bourgeoisie, resulting in a period of growth of cities and commerce. At this time, problems in demography, insurance business, observational errors and many statistical problems which arose as a result of the development of capitalistic relations and presented a decisive stimulus for the birth of probability. The development of the capitalistic system, with its monetary form of exchange, led to games of chance becoming a mass phenomenon with analogous problems raised in other fields of human endeavour behind them.

From a mathematical point of view, the birth of probability coincided with the development of analytic geometry, differential and integral calculus and combinatorics. Up to the middle of the 17^{th} century, no general method for solving probabilistic problems was available. There were many materials resulting from various branches of human activity related to probabilistic topics, but a theory of probability had not being created yet. To exemplify, back in the 16^{th} century, Cardano was able to calculate the number of possible outcomes with and without repetition in the case of two and three dice throws. He approached the notion of statistical regularity and came close to a definition of probability in terms of the ratio of equal probability events using the idea of mathematical expectation. Around mid-17^{th} century, Pascal, Fermat and Huygens applied the

addition[1] and multiplication[2] rules of probability and were familiar with the notions of dependence and independence of events and mathematical expectation. However, these ideas were developed only in the simplest cases, and appeared as solutions to particular problems rather than going further into the development of concepts and rules as general statements [46].

Even though there remains to be a general consensus in mathematics, in this thesis we consider Bernoulli and De Moivre as the founders of classical probability. Both these authors acknowledged the works of Cardano and Tartaglia, Fermat, Pascal, Huygens and Montmort among others in their books, but they came up with a greater level of generality. Unlike Cardano, Bernoulli was able to define probability as a ratio, and De Moivre saw that the results achieved in Montmort's work may be derived from a general theorem. What leads us to crediting Bernoulli and De Moivre with creating the foundation for probability as a science is the fact that they were the first to define probability and expectation with a greater level of generality. Also, while Bernoulli presented and proved with complete analytic rigour the first convergence theorem in probability, De Moivre was aware of the general results that before him were applied to only specific problems [30].

Now that we have briefly discussed the origins of classical probability and mentioned the main authors of that period, we present the *Ars Conjectandi* of Bernoulli and his definition of probability as a ratio of favourable to possible cases. This definition became the classical standard one used from the beginning of the 18^{th} century until the rupture with the classical approach in 1933 with Kolmogorov's axioms of modern probability.

4.2.2 Bernoulli's *Ars Conjectandi* and the definition of probability

Jacques Bernoulli, also known as Jacob and James, was born in Basel, Switzerland, in 1654 and died in 1705. Bernoulli received his Master of Arts in philosophy in 1671, a licentiate in theology in 1676 and studied mathematics and astronomy. In his works, he made many contributions to calculus, and is one of the founders of calculus of variations. However, his greatest contribution was in the field of probability, where he derived the first version of the law of large numbers in his work *Ars Conjectandi* [26].

Bernoulli's book "*Ars Conjectandi*" (The art of Conjecturing), was published eight years after his death by his nephew Nicholas Bernoulli. This book played such a significant role in the history

[1]The addition rule can be stated as: $P(A \cup B) = P(A) + P(B)$ if A and B can't both happen simultaneously.
[2]The multiplication rule can be stated as: $P(A \cap B) = P(A) \cdot P(B|A)$.

of probability, that thanks to this work, probability began a new era in its development and was raised to the status of a science.

Ars Conjectandi is divided into four parts. The first one, "*A Treatise on Possible Calculations in a Game of Chance of Christian Huygens with J. Bernoulli's Comments*", consists of a reprint of Huygens work (*De Ratiocinniis in Ludo Aleae*) accompanied by Bernoulli's comments in all but one proposition. In his commentary on the 12^{th} proposition, he establishes the result known as Bernoulli's formula for the binomial distribution.

The second part of the book is called "*The Doctrine of Permutations and Combinations*". Having one entired part of his book dedicated to combinatorics gives an evidence to the extent of the usage this discipline as a basic tool for probability before the introduction of infinitesimal analysis.

The third part is called "*Applications of the Theory of Combinations to Different Games of Chance and Dicing*". He presents 24 problems, some of them solved in their general form rather than through a numerical approach. Even though these three parts made a significant contribution not only to probability, but to mathematics as a whole, the most important part of the book that marks a new era in probability history is the last one.

The fourth and last part, "*Applications of the Previous Study to Civil, Moral and Economic Problems*", was left incomplete in the sense that he didn't write about the applications in the title. This part explains his interpretation of probability and also contains the proof of Bernoulli's theorem, that is, the weak law of large numbers in its simplest form.

Regarding his definition of probability, Bernoulli states it in the classical way, as the ratio between favourable and possible outcomes. Nevertheless, he is conscious that "*... this by no means takes place with most other effects that depend on the operation of nature or on human will*". Thus, for the cases which we can't regard as equally likely to occur, or for which we can't a priori have an idea of its probability, because we don't know the number of favourable and possible outcomes, Bernoulli states we can still find the probability "*... a posteriori from the results many times observed in similar situations, since it should be presumed that something can happen or not to happen in similar circumstances in the past*" [7] (p. 326-327). However, Bernoulli calls attention to a possible misunderstanding. He mentions that the ratio we are seeking to determine through observation is only approximate, and can never be obtained with absolute accuracy. "*Rather, the ratio should be defined within some range, that is, contained within two limits, which can be made as narrow as anyone might want*" [7] (p. 329).

Following these explanations, Bernoulli goes to chapter 5 of the 4^{th} part of his book, where he states five lemmas and proves his theorem. We will present each of Bernoulli's five lemmas, as well as his *principal proposition*, or theorem, the weak law of large numbers. We will also present the ideas of Bernoulli's proofs, however using modern language and notation. The statement of each of the five lemmas and the *principal proposition* are all taken directly from *Ars Conjectandi*. [7].

4.2.3 Bernoulli's law of large numbers

Lemma 4.2.1. *Consider the two series of numbers:*

$$0, 1, 2, 3, 4, ..., r-1, r, r+1, ..., r+s$$

$$\underbrace{0, 1, 2, 3, 4, \ldots}_{A}, nr-n, \underbrace{\ldots}_{B}, nr, \underbrace{\ldots}_{C}, nr+n, \underbrace{\ldots, nr+ns}_{D}.$$

- *We can notice that the second series has n times more elements than the first one and each element of the first series can be multiplied by n and linked to an element in the second one;*
- *As we increase n, the number of terms in the parts B, C and between 0 and nr will increase;*
- *Also, no matter how large n is, the number of terms in D won't be larger than the number of terms in B times $(s-1)$ or the number of terms in C times $(s-1)$.*
- *In the same way, the number of terms in A won't be larger than the number of terms in B times $(r-1)$ or the number of terms in C times $(r-1)$.*

We will omit the demonstration of this first lemma because the reader can simply verify it by some simple arithmetic calculations.

Lemma 4.2.2. *Every integer power of a binomial $r+s$ is expressed by one more term than the number of units in the index of the power.*

In this lemma, Bernoulli meant that, when n is an integer, the expansion of $(r+s)^n$ has $n+1$ terms. This can be verified by induction.

Lemma 4.2.3. *In any power of this binomial (at least in any power of which the index is equal to the binomial $r+s=t$, or to a multiple of it, that is, $nr+ns=nt$), if some terms precede and others*

follow some term M such that the number of all the preceding terms to the number of all the following terms is, reciprocally, as s to t (or, equivalently, if in that term the numbers of dimensions of the lefters r and s are directly as the quantities r and s themselves), then that term will be the largest of all the terms in that power, and the terms nearer it on either side will be larger than the terms farther away on the same side. But this same term M will have a smaller ratio to the terms closer to it than those nearer terms (in an equal interval of terms) have to the farther terms.

The idea of the lemma is to show the binomial expansion of $(r+s)^{nr+ns}$. By lemma (4.2.2), its expansion has $nr+ns+1$ terms:

$$\underbrace{r^n t + \frac{nt}{1} r^{nt-1}s + \frac{nt(nt-1)}{1\cdot 2} r^{nt-1}s^2 + \cdots}_{ns\ terms} + M + \underbrace{\cdots + \frac{nt}{1} r s^{nt-1} + s^n t}_{nr\ terms}$$

.

Bernoulli also states that M will be the largest term, and that the terms closer to M will be larger than those farther from it. Furthermore, the ratio between consecutive terms closer to M will be smaller than the ratio of consecutive terms farther from M.[3]

Proof. Note that the coefficients of the terms equidistant from the ends are the same. To see that there are ns terms before M and nr terms after M, note that by lemma (4.2.2), the expansion has $nr+ns$ terms plus the term M. So he states that the ratio of terms preceding M by the terms after M must be the same as s/r, and this implies that we have ns terms before M and nr terms after M.

So we can say that

$$\begin{aligned} M &= \frac{nt(nt-1)(nt-2)\cdots(nt-ns+1)}{1\cdot 2\cdot 3\cdot 4\cdots ns} r^{nr}s^{ns} = \frac{nt(nt-1)(nt-2)\cdots(nr+1)}{1\cdot 2\cdot 3\cdot 4\cdots ns} r^{nr}s^{ns} \\ &= \frac{nt(nt-1)(nt-2)\cdots(nt-nr+1)}{1\cdot 2\cdot 3\cdot 4\cdots nr} r^{nr}s^{ns} = \frac{nt(nt-1)(nt-2)\cdots(ns+1)}{1\cdot 2\cdot 3\cdot 4\cdots nr} r^{nr}s^{ns} \end{aligned}$$

We can express the two neighbours of M on the left and right in the binomial expansion as:

[3] The same is valid for non-consecutive terms. For example, the ratio between the 3^{rd} and the 6^{th} term from M will be larger than that of of the 10^{th} and the 13^{th} term from M.

$$\frac{nt(nt-1)(nt-2)\cdots(nr+3)}{1\cdot 2\cdot 3\cdot 4\cdots(ns-2)}r^{nr+2}s^{ns-2}+\frac{nt(nt-1)(nt-2)\cdots(nr+2)}{1\cdot 2\cdot 3\cdot 4\cdots(ns-1)}r^{nr+1}s^{ns-1}+M$$
$$+\frac{nt(nt-1)(nt-2)\cdots(ns+2)}{1\cdot 2\cdot 3\cdot 4\cdots(nr-1)}r^{nr-1}s^{ns+1}+\frac{nt(nt-1)(nt-2)\cdots(ns+3)}{1\cdot 2\cdot 3\cdot 4\cdots(nr-2)}r^{nr-2}s^{ns+2}$$

Now we divide the neighbours as per the items bellow and we can draw the conclusion of the lemma:

(1) Dividing M by the term on its left, we get: $\frac{(nr+1)s}{ns\cdot r}$ and $(nr+1)s > ns\cdot r$, which implies M is bigger than its left neighbor.

(2) Dividing the first M left neighbor by the next left neighbor we get: $\frac{(nr+2)s}{(ns-1)r}$ and $(nr+2)s > (ns-1)r$, so the first left neighbor is greater than the second one.

(3) Dividing M by the term on its right, we get: $\frac{(ns+1)r}{nr\cdot s}$ and $(ns+1)r > nr\cdot s$, so M is bigger than its right neighbor.

(4) Dividing the first M right neighbor by the next right neighbor we get: $\frac{(ns+2)r}{(nr-1)s}$ and $(ns+2)r > (nr-1)s$, so the first right neighbor is greater than the next one.

Doing this procedure recursively, we can figure out that M is the greatest element in the expansion and the elements reduce as they get farther from M.

We can also notice that $\frac{(nr+1)s}{ns\cdot r} < \frac{(nr+2)s}{(ns-1)r}$ and that $\frac{(ns+1)r}{nr\cdot s} < \frac{(ns+2)r}{(nr-1)s}$. So doing this procedure recursively we can see that M has smaller ratios to nearer terms than to further ones on the same side. □

Lemma 4.2.4. *In a power of a binomial with index nt, the number n can be conceived to be so large that the largest term M acquires a ratio to the terms α and β, which are at an interval of n terms to the left and right of it that is larger than any given ratio.*

The goal of this lemma is to show that: $\lim_{n\to\infty}\frac{M}{\alpha}=\infty$ and $\lim_{n\to\infty}\frac{M}{\beta}=\infty$.

Proof.

$$M=\frac{nt(nt-1)(nt-2)\cdots(nr+1)}{1\cdot 2\cdot 3\cdot 4\cdots ns}r^{nr}s^{ns}=\frac{nt(nt-1)(nt-2)\cdots(ns+3)}{1\cdot 2\cdot 3\cdot 4\cdots(nr-2)}r^{nr-2}s^{ns+2}$$

On the left,

$$\alpha = \frac{nt(nt-1)(nt-2)\cdots(nr+n+1)}{1\cdot 2\cdot 3\cdot 4\cdots(ns-n)} r^{nr+n} s^{ns-n}$$

On the right,

$$\beta = \frac{nt(nt-1)(nt-2)\cdots(ns+n+1)}{1\cdot 2\cdot 3\cdot 4\cdots(nr-n)} r^{nr-n} s^{ns+n}$$

Now we can put the ratios:

$$\begin{aligned}\frac{M}{\alpha} &= \frac{(nr+n)(nr+n-1)(nr+n-2)\cdots(nr+1)s^n}{(ns-n+1)(ns-n+2)(ns-n+3)\cdots ns\cdot r^n} \\ &= \frac{(nrs+ns)(nrs+ns-s)(nrs+ns-2s)\cdots(nrs+s)}{(nrs-nr+r)(nrs-nr+2r)(nrs-nr+3r)\cdots nrs}\end{aligned}$$

$$\begin{aligned}\frac{M}{\beta} &= \frac{(ns+n)(ns+n-1)(ns+n-2)...(ns+1)r^n}{(nr-n+1)(nr-n+2)(nr-n+3)...nr\cdot s^n} \\ &= \frac{(nrs+nr)(nrs+nr-r)(nrs+nr-2r)...(nrs+r)}{(nrs-ns+s)(nrs-ns+2s)(nrs-ns+3s)...nrs}\end{aligned}$$

As n goes to infinity, the numbers $(nr \pm n \pm 1), (nr \pm n \pm 2), ...$ and the numbers $(ns \pm n \pm 1), (ns \pm n \pm 2), ...$ will all have the same values of $(nr \pm n)$ and $(ns \pm n)$. Now we can say that:

$$\frac{M}{\alpha} = \frac{(rs+s)(rs+s)...rs}{(rs-r)(rs-r)...rs}$$

As we have n factors both in the numerator and in the denominator, we have that: $\frac{M}{\alpha} = \left(\frac{rs+s}{rs-r}\right)^n$, which is an infinitely large value. Similarly, we have that $\lim_{n\to\infty} \frac{M}{\beta} = \infty$. □

Lemma 4.2.5. *Given what has been posited in the preceding lemmas, n can be taken to be so large that the sum of all the terms between the middle and maximum term M and the bounds α and β inclusively has to the sum of all the remaining terms outside the bounds α and β a ratio larger than any given ratio.*

In other words, Bernoulli is stating that the ratio of the sum of all terms from α up to β to the sum of all the remaining terms may be made arbitrarily large as n increases.

Proof. Out of the terms between M and the bound α. Let's call the second term from the maximum F, the third G, the fourth H and so on, and let the first term to the left of α be called P, the second

one Q, the third R. So the terms could be placed like: $...R, Q, P, \alpha, ..., H, G, F, M, ...$ Now, from lemma (4.2.3) we have: $\frac{M}{F} < \frac{\alpha}{P}; \frac{F}{G} < \frac{P}{Q}; \frac{G}{H} < \frac{Q}{R}$ and so forth. We can also conclude that $\frac{M}{\alpha} < \frac{F}{P} < \frac{G}{Q} < \frac{H}{R}$ and so successively.

From lemma (4.2.4), $n \to \infty \Rightarrow \frac{M}{\alpha} \to \infty$ as do the fractions $\frac{F}{P}; \frac{G}{Q}; \frac{H}{R}...$. So we can conclude that $n \to \infty \Rightarrow \frac{F+G+H+...}{P+Q+R+...} \to \infty$. So the sum of the terms between M and α is infinitely larger than the sum of the same number of terms to the left of α. But by the lemma (4.2.1), the number of terms to the left of α doesn't exceed $(s-1)$ times the number of terms between M and α, that is, a finite number of times. Also, by lemma (4.2.3), the terms become smaller as they approach the extremes, that is, farther to the left of α. We can see that the sum of the terms between M and α will be infinitely larger than the sum of all the terms beyond α. The same can be said about the terms between M and β. Finally, the sum of the terms between α and β will be infinitely larger than the sum of all the other terms. □

After this proof, Bernoulli also presents an alternative way of proving the lemmas (4.2.4) and (4.2.5) because he was concerned about the reception of the idea that when n goes to infinity, the numbers $(nr \pm n \pm 1), (nr \pm n \pm 2), ...$ and the numbers $(ns \pm n \pm 1), (ns \pm n \pm 2), ...$ will all have the same values of $(nr \pm n)$ and $(ns \pm n)$, as presented in lemma (4.2.4).

Bernoulli shows that for any given (large) ratio c, we can find a finite n such that the ratio of the sum of the terms between the bounds α and β to all the other terms (the terms in the queue) will be larger than c.

So for any value c, we can find a finite n such that if we take the binomial $(r+s)^n$ with its terms represented as:

$$\underbrace{a, \cdots, f, g, h}_{n(s-1)\ terms}, \underbrace{\boldsymbol{\alpha}, \cdots, F, G, H}_{n\ terms}, \mathbf{M}, \underbrace{U, V, W, \cdots, \boldsymbol{\beta}}_{n\ terms}, \underbrace{u, v, w, \cdots, z}_{n(r-1)\ terms},$$

it is true that $\frac{\boldsymbol{\alpha}+...+F+G+H+M+U+V+W+...+\boldsymbol{\beta}}{a+...+f+g+h+u+v+w+...+z} > c$.

To show this result, let's take a ratio which is smaller than $\frac{rs+s}{rs-r}$. For example, we can take $\frac{r+1}{r} = \frac{rs+s}{rs} < \frac{rs+s}{rs-r}$. Now, we multiply this ratio $\frac{r+1}{r}$ by itself as many times as necessary to make it greater than or equal to $c(s-1)$, say k times, so we get $\left(\frac{r+1}{r}\right)^k \geq c(s-1)$.

Now, looking at the ratio $\frac{M}{\alpha} = \frac{(nrs+ns)}{(nrs-nr+r)} \cdot \frac{(nrs+ns-s)}{(nrs-nr+2r)} \cdot \frac{(nrs+ns-2s)}{(nrs-nr+3r)} \cdots \frac{(nrs+s)}{nrs}$, each individual fraction is less than $\frac{(rs+s)}{(rs-r)}$, but each of these individual fractions approaches $\frac{(rs+s)}{(rs-r)}$ as n

increases.

Then we can see that among these fractions, the product of which gives $\frac{M}{\alpha}$, one of them will be $\frac{rs+s}{rs}$ or equivalently $\frac{r+1}{r}$. Let's find the value of n such that the fraction in the k^{th} position will be equal to $\frac{r+1}{r}$.

$$\frac{M}{\alpha} = \underbrace{\frac{(nrs+ns)}{(nrs-nr+r)}}_{1^{st}\ position} \cdot \underbrace{\frac{(nrs+ns-s)}{(nrs-nr+2r)}}_{2^{nd}\ position} \cdot \underbrace{\frac{(nrs+ns-2s)}{(nrs-nr+3r)}}_{3^{rd}\ position} \cdots \underbrace{\frac{nrs+ns-ks+s}{nrs-nr+kr}}_{k^{th}\ position} \cdots \underbrace{\frac{(nrs+s)}{nrs}}_{n^{th}\ position}$$

The fraction in the k^{th} position is $\frac{nrs+ns-ks+s}{nrs-nr+kr}$. Now we find n by: $\frac{nrs+ns-ks+s}{nrs-nr+kr} = \frac{r+1}{r} \Rightarrow n = k + \frac{ks-s}{r+1}$ and $nt = kt + \frac{kst-st}{r+1}$.

We will show that when the binomial $(r+s)$ is raised to the power $nt = kt + \frac{kst-st}{r+1}$, the maximum term M will exceed the bound α more than $c(s-1)$ times, that is $M > \alpha c(s-1)$, or $\frac{M}{\alpha} > c(s-1)$.

Too see this, note that the fraction in the k^{th} position raised to the power k is, by construction, greater than $c(s-1)$, that is: $(\frac{r+1}{r})^k > c(s-1)$.

The fraction in the preceding positions are all greater than the one in the k^{th} position, then

$$\underbrace{\frac{(nrs+ns)}{(nrs-nr+r)}}_{1^{st}\ pos} \cdot \underbrace{\frac{(nrs+ns-s)}{(nrs-nr+2r)}}_{2^{nd}\ pos} \cdots \underbrace{\frac{r+1}{r}}_{k^{th}\ pos} > \underbrace{\frac{r+1}{r} \cdot \frac{r+1}{r} \cdots \frac{r+1}{r}}_{k\ times} = (\frac{r+1}{r})^k > c(s-1)$$

and we can conclude that the product of all the individual fractions will be even greater, so we can say that: $\frac{M}{\alpha} > c(s-1)$.

Looking at the expansion of the binomial, by lemma (4.2.3), we can say that $\frac{M}{\alpha} < \frac{H}{h} < \frac{G}{g} < \frac{F}{f}$ and so successively, until the ratio of the last term in the bound, α, and its correspondent term outside the bound (the n^{th} term to the left of α) that we will call d_α. Now, $M > \alpha c(s-1)$ implies that $H > hc(s-1), G > gc(s-1), F > fc(s-1), ..., \alpha > d_\alpha c(s-1)$. Now, summing the terms in the left and in the right of these inequalities yields:

$$\begin{aligned} H+G+F+...+\alpha &> hc(s-1)+gc(s-1)+fc(s-1)+...+d_\alpha c(s-1) \\ &= (h+g+f+...+d_\alpha)c(s-1) \end{aligned}$$

Finally, as we have n terms inside the bound and $n(s-1)$ terms in the left tail, we can conclude

that: $M + H + G + F + ... + \alpha > c(h + g + f + ... + a)$ or $\frac{M+H+G+F+...+\alpha}{h+g+f+...+a} > c$.

Bernoulli develops the same argument for the terms on the right side of M, and finds that the n multiplied by t that will accomplish this task is: $kt + \frac{krt-rt}{s+1}$. So taking the maximum between this term and $kt + \frac{kst-st}{r+1}$ we can conclude, finally, that: $\frac{\boldsymbol{\alpha}+...+F+G+H+M+U+V+W+...+\boldsymbol{\beta}}{a+...+f+g+h+u+v+w+...+z} > c$.

Now that all the 5 lemmas have been demonstrated, Bernoulli finally presents his *principal proposition*, which is stated and demonstrated below. Just a brief clarification of the language used: what Bernoulli called *fertile cases* is equivalent to favourable cases in today's terminology, with *sterile cases* being the complement of the former.

Theorem 4.2.6. *Let the number of fertile [or favourable] cases and the number of sterile [or non favourable] cases have exactly or approximately the ratio r/s, and let the number of fertile cases to all the cases be in ratio $\frac{r}{r+s}$ or r/t, which ratio is bounded by the limits $\frac{r+1}{t}$ and $\frac{r-1}{t}$. It is to be shown that so many experiments can be taken that it becomes any given number of times (say c times) more likely that the number of fertile observations will fall between these bounds than outside them, that is, the ratio of the number of fertile to the number of all the observations will have a ratio that is neither more than $\frac{r+1}{t}$ nor less than $\frac{r-1}{t}$.*

Proof. Let's consider nt to be the number of observations.

The probability of having $0, 1, 2, 3, \ldots$ failures is expressed by:

$$\frac{r^{nt}}{t^{nt}}, \quad \frac{nt}{t^{nt} \cdot 1} r^{nt-1} s, \quad \frac{nt(nt-1)}{t^{nt} \cdot 1 \cdot 2} r^{nt-2} s^2, \quad \frac{nt(nt-1)(nt-2)}{t^{nt} \cdot 1 \cdot 2 \cdot 3} r^{nt-3} s^3, \quad \ldots$$

Doing this procedure recursively, we can see that these are the terms in the expansion of the binomial $(r + s)$ raised to the power nt divided by t^{nt}. Furthermore, the probability of having nr favourable cases and ns non favourable cases is represented by the term M in the binomial expansion (divided by t^{nt}), and the probability of having $nr + n$ or $nr - n$ favourable cases is associated to the bounds α and β.

The sum of the cases for which we have not more than $nr + n$ and not less than $nr - n$ favourable occurrences is expressed by the sum of the terms of the power contained between the bounds α and β.

The power of the binomial can be taken to be great enough, so the sum of the terms included between the bounds α and β exceeds more than c times the sum of the terms in the tail. So we

can take a large number of observations such that **the sum of the cases** in which the ratio of the number of favourable observations to the total number of observations will be between $\frac{nr-n}{nt}$ and $\frac{nr+n}{nt}$ (or equivalently $\frac{r+1}{t}$ and $\frac{r-1}{t}$), **will exceed the sum of the remaining cases by more than c times**.

In Bernoulli's own words, *"it is rendered more than c times more probable that the ratio of the number of fertile observations to the number of all the observations will fall within the bounds $\frac{r+1}{t}$ and $\frac{r-1}{t}$ than that it will fall outside*" (p. 338-339). □

After this demonstration, Bernoulli gives an example where he gives values to r, s and c and he finds the total number of observations n according to his theorem.

In his example he sets: $r = 30$, $s = 20$, $t = r + s = 50$ and $c = 1000$.

To the left side, $$(\frac{r+1}{r})^k \geq c(s-1) \Rightarrow k \geq \frac{\log[c(s-1)]}{\log(r+1) - \log r} = \frac{4.2787536}{142405} = 301.$$
$$nt = kt + \frac{kst - s}{r+1} < 24,728.$$

To the right side, $$(\frac{s+1}{s})^k \geq c(r-1) \Rightarrow k \geq \frac{\log[c(r-1)]}{\log(s+1) - \log s} = \frac{4.4623980}{211893} = 211.$$
$$nt = kt + \frac{krt - r}{s+1} < 25,550.$$

If 25,550 trials were performed, it will be more than 1000 times more likely that the ratio of favourable to the total number of observations will be between the bounds: 31/50 and 29/50 than outside these bounds.

In modern notation, Bernoulli's theorem can be stated as: if the probability of occurrence of an event A in a sequence of n independent trials is p, and the total number of favourable cases is m, then for any positive ε, one can assert with probability as close to 1 as desired, that for a sufficiently large number of trials n, the difference $m/n - p$ is less than ε in absolute value: $P\{|m/n - p| < \varepsilon\} > 1 - \eta$, where η is an arbitrarily small number [46] (p. 74).

In this case, m/n is the empirical result of the trials and p is the M^{th} term in the binomial expansion. So the difference between the estimation and the true probability measure could be

made arbitrarily small by raising the number of Bernoulli trials.

We need to clarify here that in Bernoulli's theorem, no matter how large we choose n to be, it is still possible to find instances in a sequence of n trials in which the difference $|(m/n) - p|$ is greater than ε. However, Bernoulli's theorem guarantees that for n sufficiently large, in the majority of cases, the inequality $|(m/n) - p| < \varepsilon$ will be satisfied (or we can say that the set of divergent points has measure zero) [46] (p. 74).

Hald [30] (p. 263) mentions that Bernoulli's theorem is very important for probability theory, because it gives a theoretical and rigorous justification for the usage of an estimator for a probability, however, it doesn't say how to find an interval for the probability p from an observed value of m/n because the total number of observations depends on p, t and c. On the other hand, Maistrov [46] (p. 75) argues that the theorem doesn't state that $\lim_{n\to\infty} m/n = p$ rather, it states that the probability of large deviations of the frequency m/n from the probability p is small, if the number of trials n is large enough.

4.2.4 De Moivre's work - *The Doctrine of Chances*

Abraham de Moivre was born in Vitry-le-François, France, in 1667 and died in London, in 1754. He was one of the many gifted Protestants who emigrated from France to England. While his formal education was in French, his many contributions were made within the Royal Society of London. His father, a provincial surgeon of modest means, assured him of a competent but undistinguished classical education. He read mathematics almost in secret, and Christiaan Huygens' work on the mathematics of games of chance, *De ratiociniis in ludo aleae*, formed part of this clandestine study [26].

He dedicated his masterpiece, *The Doctrine of Chances*, to his friend Newton, and this book became the standard knowledge of probability at that time. Among his contributions, we can list his approximation to the binomial probability distribution. Bernoulli proved the weak law of large numbers, and De Moivre's approximation to the binomial distribution was conceived as an attempt to improve this result. Bernoulli did some numerical examples of a binomial approximation for particular values of n and p, but De Moivre was able to state the approximation to the binomial distribution in a more general way.

As mentioned before, along with Bernoulli's work, De Moivre's work is also of crucial importance, because the concepts they developed with attained a degree of generality that raised

probability theory to the status of science. De Moivre's book, *The Doctrine of Chances*, brings a definition of probability, some elementary theorems and some important advances in probability techniques. For example, it improved the ways of calculating tails of binomial probabilities brought by Bernoulli, which led to new proofs of the law of large numbers, and precise statements for local and for integral limiting theorems [59]. However, for the purpose of this thesis we are interested in the definition of probability and in the theorems of total probability (or the addition theorem) and compound probability (or the multiplication theorem).

Just like Bernoulli, De Moivre defines probability as the ratio of favourable to possible outcomes. In his own words: "*if we constitute a fraction whereof the numerator be the number of chances whereby an event may happen, and the denominator the number of all chances whereby it may either happen or fail, that fraction will be a proper designation of the probability*"[21] (p. 1). In the introduction, De Moivre also defines the expectation of a player's prize as his probability of winning times the value of the prize.

Regarding the theorems of addition and multiplication, De Moivre states that if two events are independent and the first has probability of success p and failure q, and the second one has probability of success r and failure s, then the product $(p+q)\cdot(r+s) = pr + qr + ps + qs$ contains all the chances of success and failure of both events. This is known as the multiplication rule for independent events, which also implies the addition rule. De Moivre also says that this method may be extended to any number of events, and he derives a binomial distribution through the problems he resolves in his book. He does not discuss the multiplication rule in a general way for dependent events in the introduction, but many of his problems lead to drawings without replacement from a finite population. To those cases, he uses the multiplication rule adjusting the conditional probabilities *ad hoc*. The case for dependent events was treated independently by Bayes in 1764[4] and Laplace in 1774 [30]. This case will be the object of our focus, drawing primarily from Bayes' contributions.

4.2.5 Bayes' contribution

Thomas Bayes was born in London, in 1702, and died in Tunbridge, Wells, in 1761, and, just like his father, he was a theologian. The Royal Society of London elected him a fellow in 1742 [26]. One clarification on Bayes here is necessary. In this thesis, we are not interested in discussing the

[4] 1764 is the year of the posthumous publication.

Bayesian interpretation of probability, as this goes beyond the scope of our work. We are interested here in Bayes' developments for the probability of dependent events, or the theorem that takes his name.

In his work called *An Essay towards solving a Problem in the Doctrine of Chances*, Bayes developed the binomial distribution's curve and established a rule for obtaining an interval for the probability of an event, assuming a uniform prior distribution of the binomial parameter p. After observing m successes and n failures, $P(a < p < b|m, n) = \frac{\int_a^b \binom{m+n}{m} p^m (1-p)^n dp}{\int_0^1 \binom{m+n}{m} p^m (1-p)^n dp}$.

In this thesis, however, we will concern ourselves with his results in conditional probability that imply the theorem that carries his name, and deal with the product rule for dependent events. After the definition of *probability* and *expectation* from Bernoulli and De Moivre, Bayes' developments in *conditional probability* is the key element that was missing in the theoretical scope of classical probability.

In Bayes' own words, *"If there be two subsequent events, the probability of the second $\frac{b}{N}$ and the probability of both together $\frac{P}{N}$, and it being first discovered that the second event has also happened, the probability I am right is $\frac{P}{b}$"* [6] (p. 381).

In today's notation, we could say that: $P(B|A) = \frac{P(A \cap B)}{P(A)}$, if $P(A) \neq 0$, which implies i) $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$ if $P(B) \neq 0$ and also implies ii) the notion of independence, because: $P(A \cap B) = P(A)P(B|A) = P(A)P(B)$ when A and B are independent, that is, the occurrence of one doesn't affect the occurrence of the other.

4.2.6 Paradoxes in classic probability

At the beginning of the 19^{th} century, geometric probability was incorporated in to the classical theory and instead of counting equally likely cases, their geometric extension (area or volume) was measured. Nevertheless, probability remained seen as a ratio, even at the beginning of the 20^{th} century, when measure theory was created and the class of sets on which we can define a geometric measure was broadened. Shafer and Vovk [56] say that a reader from the 19^{th} century would have seen nothing new if he could see the definition of probability from a measure theoretic book from the beginning of the 20^{th} century. To finish this section on classical probability, we discuss some paradoxes that were sources of dissatisfaction with the classical approach.

These paradoxes put in evidence two important limitations from classic probability. The first one is the lack of rigour to define parameters and model a problem, allowing different values for

the probability of the same event to be found. The second limitation comes from the definition of probability based on equally likely cases, which is not appropriate for dealing with many situations.

The chord paradox: This paradox is found in Bertrand [8]. Bertrand was a mathematician aware of the ill-defined nature of certain probability problems that were very influential. Very often these paradoxes are called Bertrand paradoxes.

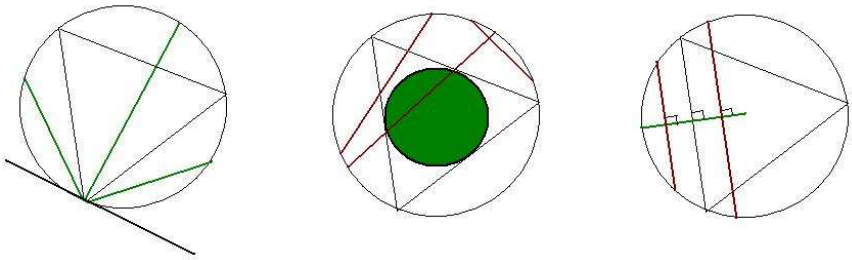


Figure 4.1: Chord paradox - [1] (p. 4).

Let's consider a disk with an inscribed equilateral triangle. What is the probability that a chord chosen at random will be longer than one of the sides of the triangle? The Figure (4.1) illustrates three possible answers for this question and the solution of the paradox concerns the way one specifies of the probability space.

Without loss of generality, let's assume that one of the two points of the chord is at the same place as one of the vertices of the triangle. The other two vertices of the triangle will split the angle formed from the first vertex with a tangent to that point on the disk in three equal parts. So we can say that 1/3 of the chords will be longer than one of the sides of the triangle.

A chord can also be determined by its midpoint. If the chord's length exceeds the side of an inscribed equilateral triangle, position it so its midpoint lies inside a smaller circle with radius one half that of the original disk. The set of favourable midpoints covers 1/4 of the original disk's area. So the proportion of favourable chords is 1/4, and not 1/3.

Another way to face this problem is by rotational symmetry. Let's fix the radius that the midpoint of the randomly selected chord will lie on. The proportion of favourable outcomes is all points on the radius that are closer to the center than half the radius, so it is 1/2.

Buffon's needle paradox: Suppose we have a large amount of lines, each of which is 10 cm apart from the other. What is the probability that a needle of 5 cm intersects with one of the lines when it's dropped on the ground?

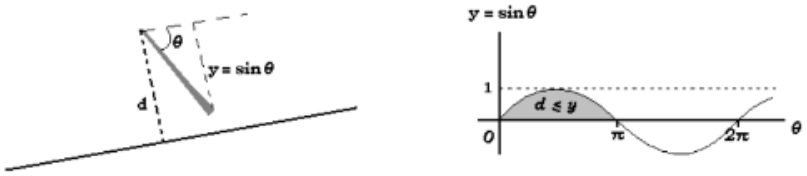


Figure 4.2: Buffon's needle paradox - [1] (p. 6).

The needle can intersect at most one line. The quantities we are interested in are the distance d of the needle's tip to the line and the angle θ that the needle forms with that line. Taking those two quantities as random and independent, the favourable outcomes are those for which $d \leq \sin\theta$. We have that $0 \leq d \leq 5$ and $0 \leq \theta \leq 2\pi$. The proportion that satisfies $d \leq \sin\theta$ is given by π^{-1}. This problem also gives a different result if we reparameterise it with $y = \sin\theta$ as shown in Figure (4.2). This paradox is discussed in [59] and the problem lies in the use of symmetry to assign probability to elementary events.

The jewelry box paradox: Suppose we have three identical jewelry boxes with two drawers in each box and one medal in each drawer. Box A has 2 golden medals, box B has 2 silver medals and box C has one golden and one silver medal. We pick up a box a random, open one drawer of that box and and look at the color of the medal inside. What is the probability that we have chosen box C?

If we randomly open one drawer from one of the three boxes and we find a golden medal, there are two possibilities: i) the other drawer of that chosen box has has another gold medal, so we have picked box A or ii) the other drawer has a silver medal, so we have picked box C. In case we find a silver medal instead of a golden one when we open the first drawer, the two possibilities are: i) the other drawer has a gold medal, so we have picked box C or ii) the other drawer has another silver medal, so we have picked box B.

Regardless of whether we have found a gold or sliver medal when we open the first drawer, one of the three boxes will have been eliminated from the problem. After seeing the first medal, we have only two options and one of these options is box C with probability 1/2.

Poincaré [51] discusses this problem on pages 26 and 27 and proposes labelling each drawer with α and β on a place we can't see the labels. By putting the secret labels, there are six equally likely cases for the drawer we open.

Box:	A	B	C
Drawer α	gold	silver	gold
Drawer β	gold	silver	silver

In case we find a gold medal in the drawer we have opened, it can be explained by three possible cases: i) box A, drawer α, ii) box A, drawer β and iii) box C, drawer α. Out of the three, only one favors the choice of box C, with probability 1/3.

Those paradoxes illustrate us two important lessons: i) that equally likely cases must be detailed enough to avoid ambiguities and ii) the need to consider the real observed event of nonzero probability that is represented in an idealized way by an event of zero probability. These two lessons were not easy for everyone, and the confusion around the paradoxes was another source of dissatisfaction with the classical approach to probability based on equally likely cases, as illustrated in chapter 2 with the epistemological obstacle of equiprobability. It will be shown in chapter 5 that Kolmogorov's approach, enables us with the concept of a probability space, where the probability measure is uniquely specified. With this approach, there is no room for ambiguities and the probability space should be carefully looked into.

4.3 The development of measure theory

The developments in measure theory, pioneered by Borel in 1898, and the further developments from Lebesgue, Radon, Carathéodory, Fréchet and Nikodym, provided a conceptual basis and opened a road towards modern probability. The ideas in this section show the evolution of the main accomplishments in measure theory that have broadened the ideas of sets and lengths, and took the notion of integral to a more general context beyond Euclidean spaces, allowing the probability axioms to be developed in a fully abstract basis.

In this section we will consider the key results of measure theory that were relevant to the development of probability. We start by an illustration with the work of Gyldén, that predated the foundation of measure theory, but soon motivated its association with probability. Following, we present Jordan's content of sets, a first work toward measure theory, but with some unconsistencies. We then present Borel and Lebesgue's work that are considered the starting point of measure theory. Borel's work is important because it broadened the type of sets that we can consider when

evaluating probabilities and Lebesgue's work is crucial because it generalized the notion of integration and allowed many convergence theorems involving limits and integrals. Carathéodory's work was important because he developed the notions of inner and outer measure and also his extension theorem is a key result that allowed a formalization of probability beyond finite sample spaces. Fréchet's contribution allowed the development of probability beyond Euclidean spaces and finally, Radon-Nikodym's theorem allowed a complete and abstract notion of the integral that allow the broadening of the concept of probability conditional to measure zero sets.

4.3.1 Gyldén's continued fractions

Von Plato [67], found that the description of the first problem that motivated the association of measure theory and probability came from the astronomer Hugo Gyldén in 1888. He was concerned about the long-term behaviour of motions of bodies, more specifically, on the convergence in the approximate computation of planetary motions. Gyldén was asking whether there exists an asymptotic mean motion. Probability entered his study through the use of continued fractions.

A continued fraction is given by taking a real number x and calling its integer part a_0, so, $x = a_0 + x_1$, with $x_1 \in [0, 1]$. We take $1/x_1$ and call its integer part a_1, so $1/x_1 = a_1 + x_2$ with $x_2 \in [0, 1]$. This process is repeated successively and the real number x can be represented as a continued fraction:

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}.$$

In this manner, a real number can be represented by a sequence, $x = (a_0, a_1, a_2, ...)$. Gyldén's question on the limiting distribution of the integers in a continued fraction was prompted by a question in the perturbation theory of planetary motions. He was asking if there exists a mean motion of a variable ω describing planetary motion. In his case, ω is given by a multiple of time ct plus a bounded function of time χ. Dividing by t we get: $\frac{\omega}{t} = (c + \frac{\chi}{t}) \to c$ as $t \to \infty$, so the constant c is the mean motion.

Gylden's frequent work involving continued fractions led him to make an observation: rational numbers are a special case of continued fractions, because, unlike the irrational numbers, their expansions terminate. Poincaré also compared rational and irrational numbers and found an important result for probability. In his 1896 book on the calculus of probability, he found that a

number is rational with probability 0, so with an infinity of possible results, probability 0 doesn't always mean impossibility, and probability 1 may not imply certainty [67] (p. 7).

Gyldén's work was reviewed by Anders Wiman's in 1900, who was the first to use measure theory with probabilistic purposes. He gave an exact determination of the limiting distribution law of a_n as n grows in the continued fractions expansions. Another important work that associated probability with physics came from Weyl in 1909-1910, who studied the distribution of real numbers motivated by perturbation calculations of planetary motions. If we take a real number x and multiply it successively by 1, 2, 3, ..., and take the decimal part, we get, with probability 1, a sequence of numbers uniformly distributed in the interval $[0, 1]$, that is, an equidistribution of the reals modulo 1. Weyl made other connections between astronomy, statistical mechanics and probability, such as his Ergodic problem, where he wanted to use the physical description of a statistical mechanical system to find its long-range behavior over time [67] (p. 9).

4.3.2 Jordan's inner and outer content

Camille Jordan was born in Lyon, in 1838, and died in Paris, in 1921. Jordan entered the *École Polytechnique* at the age of 17 and became an engineer. From 1873 until his retirement in 1912 he taught simultaneously at the École Polytechnique and the *Collège de France*. He was elected a member of the Academy of Sciences in 1881. Jordan published papers in practically all branches of the mathematics of his time. Among his contributions, we can mention his works in combinatorics and his *Cours d'Analyse*, that was first published in the early 1880's and set standards which remained unsurpassed for many years. Jordan took an active part in the movement which started modern analysis. The concept of a function of bounded variation originated with him, and he also made substantial contributions to the field of algebra [26].

Jordan was concerned with the domain of functions when working with double integrals and had extended the concept of length of intervals to a larger class of sets of real numbers using finite unions of intervals. He was not satisfied with the fact that all demonstrations from that period assumed that if a bounded domain $E \in \mathbb{R}^2$ is decomposed into $E_1, E_2, \ldots$, the sum of these parts is equal to the total extension of E, which was not evident when taking the concept of domain in full generality [34].

To improve the treatment of the domain, Jordan partitioned E into squares $E_1, E_2, \cdots$ each of side-length ρ as in Figure 4.3 and called:

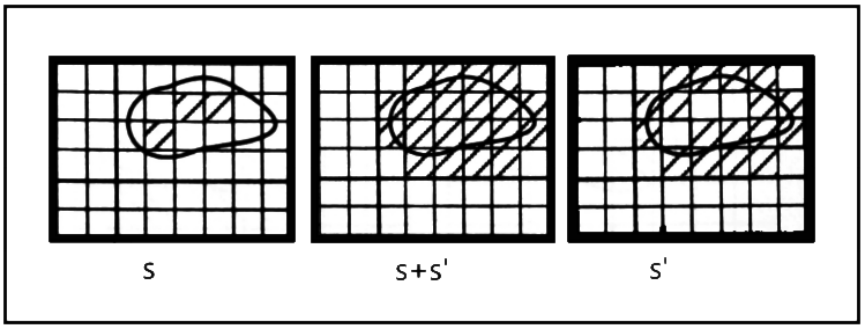


Figure 4.3: Jordan's partition - [34] (p. 276).

- S the union of the squares E_i that were interior to E;
- $S + S'$ the union of all the squares E_i that contained at least one point of E;
- S' the union of the squares that covered the boundary of E.

Then Jordan shows that we can refine the partition in such a way that $\rho \to 0$, the area S has a limit A which he called the "inner content" of E, and the sum $S + S'$ has a limit a which he called "outer content" of E. So if the limits A and a are equal, then the area of S' must vanish and E is called a *measurable* domain.

Van Dalen and Moana [64] point out that this concept of measure brings some inconveniences such as: i) there are non measurable open sets; ii) the set of rational numbers in an interval is not measurable and; iii) the measure created by Jordan is finite additive, but not countably additive.

Jordan's work was not directly related to probability, but it was an important step, along with the Borel measure, for the development of Lebesgue's measure and integral.

4.3.3 Borel and the birth of measure theory

Émile Borel was born in Saint-Affrique, France, in 1871 and died in Paris, in 1956. Borel studied at the Collège Sainte-Barbe, Lycée Louis-le-Grand and the École normale supérieure. After his graduation, Borel worked for four years as a lecturer at the University of Lille, during which time he published 22 research papers. He returned to the *École Normale* in 1897, and was appointed to the chair of theory of function, which he held until 1941 [26].

Borel extended the concept of length using countable unions when studying complex analysis in his doctoral work, *Sur quelques points de la théorie des fonctions*, in 1895 [10]. Borel's work on measure theory has a direct impact in probability. He extended the type of sets that we can evaluate

probability and also used countable additivity, which is a key concept in the axiomatization of Kolmogorov, specially when we consider infinite probability spaces.

Unlike Jordan, who was worried about the study of integrals, Borel was concerned about the convergence of complex functions on a convex curve with a dense set of divergent points. The type of functions that Borel was studying, were described by Poincaré and had the form:

$$f(z) = \sum_{n=1}^{\infty} \frac{A_n}{z - b_n}, \quad z, A_n, b_n \in \mathbb{C} \tag{1}$$

where $\sum_{n=1}^{\infty} |A_n|^{1/2} < \infty$, and $\{b_n\}$ form a subset of $C \cup S$ which is everywhere dense in C.

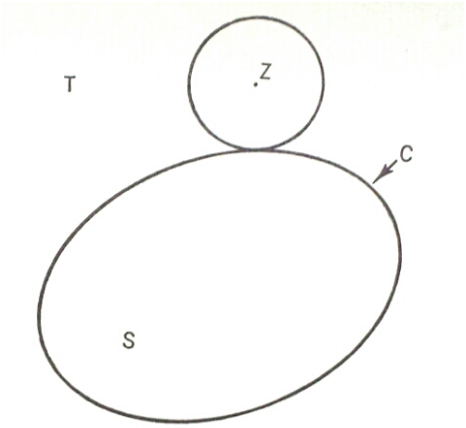


Figure 4.4: The convex curve C - [33] (p. 98).

As an illustration, in Figure 4.4, let C denote a convex contour, like a circle, that divides a plane in two regions: S, which is bounded by the contour C and T, the unbounded region. C has tangent and radius of curvature at each point, so for any point $z \in T$, there exists a circle with center z which is tangent to C and lies outside of S.

A function of the form $f(z)$ above calls the attention because it represents two distinct analytic functions: one inside and another outside the curve C and cannot be analytically continued across C. Borel [10] discovered that:

Theorem 4.3.1. *Let C denote a convex contour that divides a plane into the region S, that is bounded by C and the unbounded region T. Any point in T can be connected to any point in S by a circular arc on which the series converges, so the function can be analytically continued across C.*

This is a key result in the development of measure theory, and its proof will follow as in

Hawkins [33].

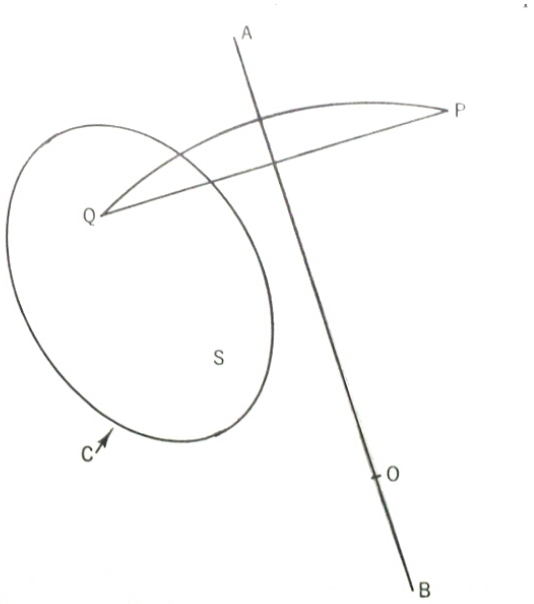


Figure 4.5: Connection of P and Q - [33] (p. 100).

Proof. Let P denote a point in T, Q be a point in S and $\overline{AB}$ denote any segment perpendicular to $\overline{PQ}$. One of the arcs $\overset{\frown}{PQ}$ can intersect the curve C at one of the points a_n. Now, suppose that for every n, the points P, Q and a_n determine a circle with center O_n lying on $\overline{AB}$ (see Figure 4.5).

If $\sum |A_n|^{1/2}$ converges, then there is another convergent series $\sum u_n$ such that $\sum |A_n|/u_n$ also converges. Now, there is an $N \in \mathbb{N}$ such that $\sum_{n=N+1}^{\infty} u_n < L/2$. So for $n > N$, we can construct intervals I_n on $\overline{AB}$ with center O_n and length $2u_n$. The sum of the lengths of I_n is $2\sum_{n=N+1}^{\infty} u_n < L$.

Now we can deduce that there are uncountably many points of $\overline{AB}$ that lie outside I_n and a point W that is not in any of the I_n, for $n = 1, 2, \cdots, N$.

As a consequence, the circle with center W that passes through P and Q contains no a_n. So it is proved that (1) converges on this circle. □

The idea of Borel's proof is based on the fact that any countable set can be covered by intervals of arbitrarily small total length. This idea is used to deduce the existence of an uncountable number of points W on $\overline{AB}$ outside I_n when $n > N$. Following this result, we have:

Corollary 4.3.1.1. *By taking a countable collection of intervals $\{I_n\}$ in $[a, b]$ with total length smaller than $b - a$, we can find an uncountable number of points in $[a, b]$ that are not in $\{I_n\}$.*

Proof. Suppose that there are only countably many points of $[a, b]$ that are not in $\{I_n\}$. So these countably many points could be covered by a countable collection of intervals $\{I_k^*\}$ of total length sufficiently small such that the total length of $\{I_k^*\}$ plus the total length of $\{I_n\}$ would still be smaller than $b - a$, a contradiction. So we conclude that we can find an uncountable number of points in $[a, b]$ that are not in $\{I_n\}$. □

As we have uncountably many points W, we have uncountably many circles that will intersect the curve C in uncountably many different points on C, for which the series (1) converges, even if the set of singularities $\{b_n\}$ is dense on C.

Borel continued to study the implications of his discovery, and in 1989 he published *Leçons sur la théorie des fonctions*, where he develops what we now call Borel sets [11]. In this thesis, instead of using the complex series mentioned before, we will concentrate on a particular case, which is the real valued series just like the approach used in [33] and [34]. For simplicity, we consider our set as the interval $[0, 1] \in \mathbb{R}$ and take the series:

$$\sum_{n=1}^{\infty} \frac{A_n}{|x - a_n|}, \quad x, a_n \in (0, 1), \quad A_n \in \mathbb{R}^+, \quad n \in \mathbb{N}, \quad A = \sum_{n=1}^{\infty} \sqrt{A_n} < \infty. \tag{2}$$

where $\{a_n : n \in \mathbb{N}\}$ is a dense set in $(0, 1)$.

If $\sum_{n=1}^{\infty} \sqrt{A_n}$ converges, then there is a series of terms u_n such that $\sum A_n/u_n$ also converges. Let's call $v_n = A_n/u_n$ and define the intervals $I_n = (a_n - v_n, a_n + v_n), \forall n \in \mathbb{N}$ and $B = \cup_{n=1}^{\infty} I_n$.

If $x \notin B$, that is, x is not in any of the intervals I_n, we have:

$$|x - a_n| > v_n \Leftrightarrow \frac{A_n}{|x - a_n|} \leq \frac{A_n}{v_n}, \quad \forall n \in \mathbb{N}.$$

We can conclude here that the total length of these intervals is $2 \sum v_n = 2v$, and the series $\sum_{n=1}^{\infty} \frac{A_n}{|x-a_n|}$ converges on $[0, 1] \backslash B$.

Now let's replace the series with terms u_n by the series with terms $u'_n = 2ku_n$, and define $v'_n(k) = A_n/u'_n$ and the intervals $I_n(k) = (a_n - v'_n, a_n + v'_n), \forall n \in \mathbb{N}$ and $B_k = \cup_{n=1}^{\infty} I_n(k)$. Then $\sum v'_n = \sum \frac{A_n}{u'_n} = \frac{1}{2k} \sum v_n$, and the series $\sum v'_n$ also converges. If $x \notin B_k$, the series (2) converges on $[0, 1] \backslash B_k$. Let D be the set of all points where the series does not converge. We have that $D \subset \cap_{k=1}^{\infty} B_k$, so $C \subset B_k$ for all $k \in \mathbb{N}$.

Once B_k consists of intervals of maximum total length $\sum_{n=1}^{\infty} 2v'_n(k) = \sum_{n=1}^{\infty} \frac{\sqrt{A_n}}{k} = \frac{A}{k}$, the

set D can be covered by countably many intervals $I_n(k), n \in \mathbb{N}$, of arbitrarily small total length by making k large enough. Therefore we can conclude that the series (2) converges on sets with measure arbitrarily close to 1 and diverges on a set D of measure 0[5].

A question that comes up at this point is: why does Borel's approach better satisfy the needs for the development of measure theory than Jordan's developments? Why is Borel considered the founder of measure theory rather than Jordan?

One possible answer to these questions lies in Jordan's use of a finite approach, that limits the needs of measure theory. Using a finite number of intervals, we can't distinguish between the set D of divergent points from the set $[0,1]\backslash D$, where the series converges. In Jordan's approach, none of these two sets are measurable, because they have inner content 0 and outer content 1. By taking a countable infinite number of intervals, Borel was able construct two disjoint measurable sets, one for the divergent points D and one for the convergent points, $[0,1]\backslash D$ [34].

Another important concept that Borel used is what is called in today's language countable additivity. Here we put it in his own words [11] with our explanation in today's notation after each part.

Lorsqu'un ensemble sera formé de tous les points compris dans une infinité dénombrable d'intervalles n'empiétant pas les uns sur les autres et ayant une longueur totale s, nous dirons que l'ensemble a pour mesure s. Lorsque deux ensembles n'ont pas de points communs, et que leurs mesures sont s et s', l'ensemble obtenu en les réunissant, c'est-à-dire leur somme, a pour mesure $s + s'$ (p. 46-47).

Borel takes a set with all of its points in countably many disjoint intervals. He says that the measure of this set, that we will denote m, is the total length s of these intervals. Also, if A_1 and A_2 are two disjoint measurable sets with $m(A_1) = s$ and $m(A_2) = s'$ then $m(A_1 \cup A_2) = m(A) + m(A_2) = s + s'$. He then immediately extends the notion of additivity of two sets to countably many sets:

"*Plus généralement, si l'on a une infinité dénombrable d'ensembles n'ayant deux à deux aucun point commun et ayant respectivement pour mesures $s_1, s_2, \ldots, s_n, \ldots$, leur somme (ou ensemble formé par leur réunion) a pour mesure $s_1 + s_2 + \cdots + s_n + \cdots$*" (p. 47).

So if $\{A_i\}, i = 1, 2, \cdots$, are countably many disjoint sets with $m(A_1) = s_1, m(A_2) = s_2, \cdots$, then $m\left(\cup_i A_i\right) = \sum_i m(A_i) = \sum_i s_i$. In the following step, he establishes the difference of two sets:

[5]Measure 0 in the sense that the set D can be covered by intervals of arbitrarily small total length.

Tout cela est une conséquence de la définition de la mesure. Voici maintenant des définitions nouvelles : si un ensemble E a pour mesure s, et contient tous les points d'un ensemble E' dont la mesure est s', l'ensemble $E - E'$, formé des points de E qui n'appartiennent pas à E', sera, dit avoir pour mesure $s - s'$; de plus, si un ensemble est la somme d'une infinité dénombrable d'ensembles sans partie commune, sa mesure sera la somme des mesures de ses parties et enfin les ensembles E et E' ayant, en vertu de ces définitions, s et s' comme mesures, et E renfermant tous les points de E', l'ensemble $E - E'$ aura pour mesure $s - s'$ (p. 47).

Here, Borel states that if $E' \subset E$ are two measurable sets with $m(E') = s'$ and $m(E) = s$, then $m(E \backslash E) = s - s'$. And finally, he concludes with the definition of countable additivity and difference of the measure of two sets and states that sets of strictly positive measure are uncountable.

La mesure de la somme d'une infinité dénombrable d'ensembles est égale à la somme de leurs mesures; la mesure de la différence de deux ensembles est égale à la différence de leurs mesures; la mesure n'est jamais négative; tout ensemble dont la mesure n'est pas nulle n'est pas dénombrable" (p. 48).

4.3.4 Lebesgue's measure and integration

Henri Léon Lebesgue was born in Beauvais, France, in 1875, and died in Paris, in 1941. He studied at the *École Normale Supérieure* from 1894 to 1897. Lebesgue had university positions at Rennes (1902—1906), Poitiers (1906—1910), Sorbonne (1910—1919), *Collège de France* (1921). In 1922 he was elected to the *Académie des Sciences*. Lebesgue's outstanding contribution to mathematics was the theory of integration that bears his name. From 1899 to 1902, while teaching at the *Lycée Centrale* in Nancy, Lebesgue developed the ideas that he presented in 1902 as his doctoral thesis at the Sorbonne. In this work Lebesgue began to develop his theory of integration which includes within its scope all the bounded discontinuous functions introduced by Baire. Although Borel's ideas of assigning measure zero to some dense sets were not welcomed by the whole community, Lebesgue accepted and completed Borel's definitions of measure and measurability so that they represented generalizations of Jordan's definitions and then used them to generalize Riemann's integral [26].

Lebesgue's concept of measure and his integral were central in the development of probability. His measure was a generalization of Jordan and Borel's measure with more interesting properties

as we will show in the following pages. Lebesgue's integral is more general than Riemman's and allows important convergence results in probability. Lebesgue also gave the concepts of measurable function and integrable functions, that are closely related to the notions of event and expectation as we will show in chapter 5.

Lebesgue continued Borel's work in measure theory, but while Borel was initially focused on the behaviour of complex series, Lebesgue, in his doctoral thesis [43], *Intégrale, Longueur, Aire* of 1902 was concerned with integration. Lebesgue discusses his famous integral in the second chapter of this thesis, but our primary focus of interest will be his first chapter, where he talks about measure of sets. After some discussion on sets and their relations, such as inclusion, he gives the definition of a measure of a set. In his own words [43] (p. 236): *Nous nous proposons d'attacher à chaque ensemble borné un nombre positif ou nul que nous apellerons sa mesure et satisfaisant aux conditions suivantes :*

(1) Il existe des ensembles dont la mesure n'est pas nulle ;

(2) Deux ensembles égaux ont même mesure ;

(3) La mesure de la somme d'un nombre fini ou d'une infinité dénombrable d'ensembles, sans points communs, deux à deux, est la somme des mesures de ces ensembles.

Under Borel's influence, Lebesgue associate the length L of an interval I to be its measure m. So $L(I) = m(I)$. And for a countable number of disjoint intervals I_n,

$$m(\sum_{n=1}^{\infty} I_n) = L(\sum_{n=1}^{\infty} I_n) = \sum_{n=1}^{\infty} L(I_n) = \sum_{n=1}^{\infty} m(I_n).$$

Lebesgue then establishes that if E is an arbitrary set and $\{I_k\}$ a countable collection of intervals (disjoint or not) and $E \subset \cup_k I_k$, it must hold: $m(E) \leq m(\cup_k I_k) \leq \sum_k L(I_k)$. So the infimum of the values of $\sum_k L(I_k)$ for coverings of E is an upper bound for a possible measure of E.

For a bounded set $E \subset \mathbb{R}$, the *outer measure* of E is given by:

$$m_e(E) = \inf \left\{ \sum_k L(I_k) : k \in \mathbb{N}, E \subset \cup_k I_k \right\}.$$

Now let $E \subset [0,1]$ and its complement $E^C = [0,1] \backslash E$. If the measure m is well defined, it is

true that $m(E^C) \leq m_e(E^C)$. So if $m(E)$ is defined, then $m(E) \geq m([0,1]) - m_e(E^C)$.

For a set $E \subset [0,1]$, the *inner measure* of E is given by:

$$m_i(E) = m([0,1]) - m_e(E^C).$$

Now Lebesgue finally defines the measurability of E as follows [43] (p. 238):

Nous appellerons ensembles mesurables ceux dont les mesures extérieure et intérieure sont égales, la valeur commune de ces deux nombres sera la mesure de l'ensembte, si le problème de la mesure est possible.

In today's notation we can say that a bounded subset $E \subset \mathbb{R}$ is called *measurable* if $m_i(E) = m_e(E)$. If this is the case, then $m(E) = m_i(E) = m_e(E)$.

Now we can ask: what is the relationship between Jordan's content and Lebesgue Measure? And what about Borel's and Lebesgue's measures?

We can say that Lebesgue generalizes both, the notion of content and Borel's measure. First, Jordan's outer content, $\overline{I}(E)$ is achieved by finite coverings while Lebesgue's outer measure, $m_e(E)$ is defined by countable coverings. It follows that $m_e(E) \leq \overline{I}(E)$. Also, as $\underline{I}(E) = 1 - \overline{I}([0,1]\backslash E)$ by taking finite intervals, and $m_i(E) = 1 - m([0,1]\backslash E)$ by taking countable intervals, we get that $\underline{I}(E) \leq m_i(E)$. The generalization comes from the fact that as $\underline{I}(E) \leq m_i(E) \leq m_e(E) \leq \overline{I}(E)$, Jordan measurable sets are a subset of the Legesgue measurable sets, or in other words, any set that is Jordan-measurable is also Lebesgue-measurable [34].

We can say that Lebesgue's measure is an extension of Borel's measure because Borel's definition doesn't guarantee that subsets of Borelian sets of measure 0 are measurable, but this statement is valid for Lebesgue's measure.

Having defined what measurable sets are, Lebesgue was able to generalize the Riemann integral. The Lebesgue integral could be applied to functions that were everywhere discontinuous. In these cases, the upper and lower Riemann sums don't converge to the same limit, so the function is not Riemann integrable.

Lebesgue [43] starts his argument introducing the definition of a "summable function" (*fonction sommable*), what we call in today's language, a function with finite integral.

Lebesgue takes a positive function f defined on the interval (a,b) and defines the set E as the region between a and b and between 0 and $f(x)$. So E is the area between the x-axis and the

positive function f defined on (a, b).

The Riemann sums s and S give the external and internal measurements of E respectively. E being Jordan-measurable is a sufficient condition for f to be integrable, and the integral is the Jordan measure of E. Lebesgue extends the definition of integral to negative functions and then he states that a summable function is a function whose integral is finite.

He starts by taking an increasing function $f(x)$ defined between α and β that takes values between a and b.

The x values are: $\alpha = x_0 < x_1 < x_2 < \ldots < x_n = \beta$.

The $f(x)$ values are: $a = a_0 < a_1 < a_2 < \ldots < a_n = b$.

The definite integral is the common limit of the two sums:

$$\sum_{i=1}^{n}(x_i - x_{i-1})a_{i-1} \qquad \sum_{i=1}^{n}(x_i - x_{i-1})a_i.$$

Donc pour définir l'intégrale d'une fonction continue croissante $f(x)$ on peut se donner les a_i, c'est-à-dire la division de l'intervalle de variation de $f(x)$, au lieu de se donner les x_i, c'est-à-dire la division de l'intervale de variation de x [43] (p. 253).

The passage above illustrates the key feature for the creation of Lebesgue's integral, which partitions the image of f and the corresponding pre-image.

Now, putting $a = a_0 < a_1 < \ldots < a_n = b$; $f(x) = a_i$ for the points of a closed set e_i, $i = 0, 1, \ldots, n$; $a_i < f(x) < a_{i+1}$, for the points of a set, sum of the intervals e_i', $(i = 0, 1, 2, \ldots, n-1)$; and the sets e_i and e_i' are measurable. As the number of a_i increases in such a way that the $\max_i\{a_i - a_{i-1}\} \to 0$, the quantities:

$$\sigma = \sum_{i=0}^{n} a_i m(e_i) + \sum_{i=1}^{n} a_i m(e_i') \quad \Sigma = \sum_{i=0}^{n} a_i m(e_i) + \sum_{i=1}^{n} a_{i+1} m(e_i')$$

go to $\int_a^b f(x)dx$ and this limit is the value of the integral.

In today's notation, a function $f : \mathbb{R} \to \mathbb{R}$ is called measurable if all the sets $\{x \in \mathbb{R} : c \leq f(x) < d\}, \quad c, d, \in \mathbb{R}, \quad c < d$, are Lebesgue measurable. If f is bounded and measurable on an interval $[a, b] \subset \mathbb{R}$, the Lebesgue integral $\int_a^b f(x)dx$ is the common limit of σ and Σ.

With Lebesgue's discovery, functions that are not Riemann integrable, such as the Dirichlet function, $f(x) = \mathbb{I}_{\mathbb{R}\setminus\mathbb{Q}}(x)$, become Lebesgue integrable.

Of course many theorems expanding this integral to negative or unbounded functions were also developed. An important contribution of this new integral to probability is the facility that it provided when taking limits in integrals using the dominated and monotone convergence theorems.

It's important to mention here the important result stated in Lebesgue [44] that is the precursor of Radon-Nikodym's theorem. He stated that any countably additive and absolute continuous[6] set function on the real numbers is an indefinite integral. Lebesgue showed that any continuous function of bounded variation has a finite derivative almost everywhere. From this point he was able to see that f being absolute continuous was a sufficient condition for having indefinite integral $F(x)$. He stated without proof that F is absolutely continuous on $[a, b]$ if and only if there exists a summable function f such that $F(x) = \int_a^x f$ for all $x \in [a, b]$. It's not hard to see that the integral F is an absolutely continuous function, but Lebesgue's great accomplishment was the ability to see the converse. Being F absolutely continuous, F has bounded variation and hence $F'(x)$ existed and was finite almost everywhere.

We can summarize these results with:

Theorem 4.3.2. *If $F(E)$ is absolutely continuous and additive, then F possesses a finite derivative almost everywhere. Furthermore, $F(E) = \int_E f(P)dP$, where $f(P)$ is equal to the derivative of F at P when this exists and $F(E)$ is equal to arbitrarily chosen values otherwise.*

4.3.5 Radon's generalization of Lebesgue's integral

Johann Radon was born in Tetschen, Bohemia (now Decin, Czech Republic), in 1887, and died in Vienna, in 1956. He entered the Gymnasium at Leitmeritz (now Litomerice), Bohemia, in 1897, and soon showed a talent for mathematics and physics. In 1905, he enrolled at the University of Vienna to study those subjects and was introduced to the theory of real functions and the calculus of variations. Radon worked through several universities in both the Czech Republic and Germany, and in 1947 obtained a full professorship at Vienna, where he spent the rest of his life. In the same year he became a full member of the Austrian Academy of Sciences.

The calculus of variations remained Radon's favorite field. He made important contributions in differential geometry, number theory, Riemannian geometry, algebra and mathematical problems

[6] Absolute continuity was a concept introduced by Vitali in 1905 [33].

of relativity theory. Radon's best-known work combined the integration theories of Lebesgue and Stieltjes and developed the concept of the integral, now known as the Radon integral [26].

It was Radon who was the first to make Lebesgue's measure theory more abstract. The idea of his generalization will be developed here following Hawkins [33]. He defined an interval in $\mathbb{R}^n$ to be the set of points $P = (x_1, x_2, \cdots, x_n)$ satisfying $a_i \leq x_i \leq b_i$, $i = 1, 2, \cdots, n$, and all sets are subsets of the interval $\mathcal{J}$, defined by $-M \leq x_i < M$, $i = 1, 2, \cdots, n$.

The class of sets T satisfies the following properties:

(1) All intervals are in T;

(2) If E_1 and E_2 are in T, then so are the sets $E_1 \cap E_2$ and $E_1 - E_2$;

(3) $\cup_{m=1}^{\infty} E_m \in T$ for all sequences (E_m) of sets $E_m \in T$.

This class contains the Borelians, and a function $f : T \to \mathbb{R}$ is called *additive* whenever $f\left(\cup_{m=1}^{\infty} E_m\right) = \sum_{m=1}^{\infty} f(E_m)$ for every sequence (E_m) of pairwise disjoint subsets of T.

Radon showed that if f is additive, then f is of bounded variation[7] and can be represented as the difference of monotone additive set functions, which means, for all $E \in T$, $f(E) = \varphi(E) - \psi(E)$.

Radon's generalization of Lebesgue's theorem (4.3.2) started by an extension of the domain for monotone functions f and introduced the notion of greatest lower bound as an analogue of the inner and outer measures. For an arbitrary set E, $\overline{f}(E)$ is the greatest lower bound of numbers of the form $\sum_i f(J_i)$, where J_i are intervals such that $E \subset \cup_i J_i$. $\underline{f}(E) = f(J) - \overline{f}(J - E)$ and E is measurable with respect to f if $\underline{f}(E) = \overline{f}(E)$.

Radon showed that the class T_1 of all f-measurable sets satisfies the three conditions mentioned above and f may be defined on T_1 by setting $f(E) = \underline{f}(E)$ or $(= \overline{f}(E))$. Given any $E \in T$ and $\varepsilon > 0$, there exists a closed set $E' \subset E$, such that $|f(E) - f(E')| < \varepsilon$. In this case, $T \subset T_1$ and f over T_1 is extended to a larger class of sets and T_1 is the natural domain of the definition of f.

If f is not monotone, we can still get the extension applying the procedure to the functions φ and ψ and the natural domain is given by the intersection of the natural domains of φ and ψ. Also, if f and T satisfy the special case given above, φ and ψ will also satisfy it and the natural domain of f contains T. Now we can make the following conclusions:

[7] A function such as f has bounded variation if, for every $E \in T$, there exists $N \in \mathbb{R}_+^*$ such that $\sum_{p=1}^{k} |f(E_p)| < N$, where (E_p), $p = 1, 2, \ldots, k$ is a finite decomposition of E [50].

- A function F is measurable with respect to f if, for every $a \in \mathbb{R}$, T_1 contains the set P such that $F(P) > a$;
- This measurable function F is summable if the series $\sum_{k=-\infty}^{\infty} a_k f(E_k)$ converges absolutely, where $\ldots < a_{-2} < a_{-1} < a_0 < a_1 < a_2 < \ldots$ is a partition of $\mathbb{R}$ with finite norm and E_k denotes the set P such that $a_k \leq F(P) \leq a_{k+1}$;
- When F is summable with respect to f, $\int_J F(P)df$ is defined to be the limit of the above series as the norm of the partition tends to zero.

We can compare these definitions with Lebesgue's work, replacing m by f, and see that Lebesgue's work becomes a particular case of Radon's[8]. In Radon, the idea of absolute continuity is not associated with Lebesgue's measure. Taking two additive set functions b and f, with natural domains T_b and T_f, b is called a basis for f if $b \geq 0$ and if $b(E) = 0$ for any set $E \subset T_b \cap T_f$, then $f(E) = 0$. When the special case applies, T_b is contained in T_f and $\forall \varepsilon > 0$, there exists a $\delta > 0$, such that $|f(E)| < \varepsilon$ whenever $b(E) < \delta$.

Radon was able to generalize Lebesgue's theorem (4.3.2) to:

Theorem 4.3.3. *If g is an additive set function with basis f, then there exists an f-summable function Ψ such that $g(E) = \int_E \Psi(P)df$ for every E in T_f.*

This theorem by Radon if the first part of the Radon-Nikodyn theormem, as we will show in this chapter, and is an important step to generalize the notions of conditional probability and conditional expectation as we will discuss in chapter 5.

4.3.6 Carathéodory's axioms for measure theory

Constantin Carathéodory was born in Berlin, in 1873, and died in Munich, in 1950. From 1891 to 1895, he attended the *École Militaire de Belgique*. After completing his education, he went to Egypt in the employ of the British government as an assistant engineer. In 1900, however, Carathéodory decided to go to Berlin to study mathematics. Carathéodory gave contributions in the calculus of variations, in the theory of functions and, in what is our main interest here, the theory of real functions, of the measure of point sets and of the integral. Carathéodory's book on

[8] Radon's work also generalizes the Stieltjes' integral. Although we will not discuss it here, as it is not the focus of this thesis, the interested reader can find an exposition of the Stieltjes integral in [59], and of Radon's generalization in [33].

this subject represents both a completion of the development begun by Borel and Lebesgue and the beginning of the modern axiomatization of this field [26].

Carathéodory introduced the concept of outer measure in $\mathbb{R}^n$ using five axioms presented in [34]:

(1) The function μ^* associates to any part of $\mathbb{R}^q$ a value in $\overline{\mathbb{R}}_+$;

(2) If $B \subset A \subset \mathbb{R}^q$, then $\mu^*(B) \leq \mu^*(A)$;

(3) If $(A_n) \subset \mathbb{R}^q$ is a finite or countable sequence of sets, $\mu^*(\cup_n A_n) \leq \sum_n \mu^*(A_n)$. A set A is measurable if it satisfies the **Carathéodory condition**, that is, for any set W, we have: $\mu^*(W) = \mu^*(W \cap A) + \mu^*(W \cap A^c)$;

(4) If $A_1, A_2 \subset \mathbb{R}^q$ and $\inf\{d(x,y) : x \in A_1, \quad y \in A_2\} > 0$, where d is the Euclidian distance in $\mathbb{R}^q$, so $\mu^*(A_1 \cup A_2) = \mu^*(A_1) + \mu^*(A_2)$;

(5) The outer measure of a set A is the $\liminf \mu^*(B)$, where B is a collection of measurable sets containing A. The inner measure of A is given by: $\mu_*(A) = \mu^*(A) - \mu^*(B \backslash A)$.

In coming up with this axiomatization of the outer measure, Caratheodory proved an important theorem that carries his name and provides us a way to extend a measure on an algebra of sets to a measure on a σ-algebra. The set of all measurable sets forms a σ-algebra and the outer measure μ^*, restricted to the set of measurable sets is a measure. Carathéodory's extension theorem is stated in many different ways, but we've chosen the version in Bartle [5]:

Theorem 4.3.4 (Carathéodory extension theorem)**.** *The collection $\mathcal{A}^*$ of all μ^*-measurable sets is a σ-algebra containing the algebra $\mathcal{A}$. Moreover, if (E_n) is a disjoint sequence of sets in $\mathcal{A}^*$, then*

$$\mu^*(\cup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu^*(E_n).$$

The idea of this theorem is that, if $\mathcal{A}$ is any algebra of subsets of a set X and if μ is a measure defined on $\mathcal{A}$, then there exists a σ-algebra $\mathcal{A}^*$ containing $\mathcal{A}$ and an outer measure μ^* defined on $\mathcal{A}^*$ such that $\mu^*(E) = \mu(E)$ for all E in $\mathcal{A}$. So the measure μ can be extended to a measure on a σ-algebra $\mathcal{A}^*$ of subsets of X that contains $\mathcal{A}$. In addition, if the measure is σ-finite, the extension is unique. This result is called the Hann extension theorem and the interested reader can consult

[4] or [53] for a complete exposition. Carathéodory's extension theorem is one of the key results in measure theory that afforded the construction of the axioms. With this extension theorem, Kolmogorov was able to take a measure defined on an algebra of sets and *extend* it to a σ-algebra generated by this algebra. This was a great result that allowed Kolgomorov create the axiom that takes probability out of the finite scope of the classical approach to infinite probability spaces.

4.3.7 Fréchet's integral on non-Euclidian spaces

Maurice Fréchet was born in Maligny, France, in 1878, and died in Paris, in 1973. At the Lycée Buffon in Paris, Frechet was taught mathematics by Jacques Hadamard, who perceived his pupil's precocity. Fréchet entered the *École Normale Supérieure* in 1900, graduating in 1903. He wrote up the lectures of Émile Borel that were turned into a book. This work was part of a long and close relationship between Borel and Fréchet that continued as long as Borel lived.

From 1907 to 1910, Fréchet held teaching positions at lycées in Besançon and Nantes and at the University of Rennes. He held a professorship at Poitiers, but was on leave in military service throughout World War I, mainly as an interpreter with the British army. From 1919 to 1928, he was head of the Institute of Mathematics at the University of Strasbourg.

Fréchet made many contributions to topology, developing the concept of metric space, compactness, separability, and completeness. He established the connection between compactness and what later came to be known as total boundedness. He came up with a great number of generalizations in topology and in Euclidian spaces and probability. Among his results, we are mainly interested in here is the formulation of an important generalization of the work of Radon, showing how to extend the work of Lebesgue and Radon to the integration of real functions on an abstract set without a topology, using merely a generalized measurelike set function [26].

Fréchet was able to take Radon's integral and raise it to a higher level of abstraction. In his work of 1959 [24], he stated Radon's integral $\int F(P)dh(P)$, where $F(P)$ is a function of a point P of an n-dimensional space, and $h(P)$ is a function of limited variation. He then proposes to state Radon's integral as $\int_E F(P)df(e)$, where $f(e)$ is an additive function of the variable subset $e \subset E$ and E is an abstract set. In Fréchet's words, *... la définition et les propriétés de l'intégrale de M. Radon s'étendent bien au delà du Calcul intégral classique, elles sont presque immédiatement applicable au domaine infiniment plus vaste du calcul fonctionnel* (p. 249). Fréchet had mentioned that in order to get this generalization, we can preserve most of Radon's definition and neglect the

nature of P, that is, a point in the n-dimensional space. By doing so, we can get an integral in the more general scope of the functional calculus.

Fréchet defines an abstract set as one that we don't know the nature of its elements, that is, this nature doesn't interfere in our reasoning regarding this set. He follows with other definitions such as a family of additive sets, a set function, total variation and limit of a sequence of sets to enter the integration of an abstract functional. He defines the upper and lower integral of a bounded functional and states that it is integrable if its upper and lower integral coincide. He then extends this integration to unbounded functionals, exposes some properties of this new integral and finishes with a section on measurable functionals.

As acknowledged by Kolmolgorov, Fréchet's integral opened paths to achieve a general and abstract axiomatization of probability. In the preface, Kolmogorov writes: *After Lebesgue's investigations, the analogy between the measure of a set and the probability of an event, as well as between the integral of a function and the mathematical expectation of a random variable, was clear. This analogy could be extended further; for example, many properties of independent random variables are completely analogous to corresponding properties of orthogonal functions. But in order to base probability theory on this analogy, one still needed to liberate the theory of measure and integration from the geometric elements still in the foreground with Lebesgue. This liberation was accomplished by Fréchet* [39] (p. v).

4.3.8 The Radon-Nikodym theorem

Although Lebesgue's integral became more general with Fréchet, who extended it to non-Euclidean spaces, the complete abstraction was accomplished by Nikodym, giving the theorem known as the Radon-Nikodym theorem. This theorem is in the heart of the modern definitions of conditional probability and conditional expectation with regards to a σ-algebra as we will show in the next chapter. In order to describe this theorem, we introduce three important definitions from [5] and [53]:

If there exists a sequence (E_n) of sets in the σ-algebra and $X = \cup E_n$ and such that $\mu(E_n) < \infty$ for all n, then we say that μ is *σ-finite*.

For example, the Lebesgue measure on $\mathbb{R}$ with the Borelian σ-algebra is not finite, but it is σ-finite. As another example, let $\mathbb{N}$ be the set of natural numbers and $\mathcal{A}$ be the σ-algebra of all the subsets of $\mathbb{N}$. If E is any subset of $\mathbb{N}$, define $\mu(E)$ to be the number of elements of E if E is finite

and equal to $+\infty$ if E is infinite. Note that μ is a measure and is called the counting measure on $\mathbb{N}$. μ is not finite, but it is σ-finite.

To exemplify a measure that is not σ-finite, think of X as a non-empty set and $\mathcal{A}$ the σ-algebra of all subsets of X. Let's define $\mu(\emptyset) = 0$ and $\mu(E) = +\infty$ if $E \neq \emptyset$.

A proposition holds *μ-almost everywhere* if there exists a subset N in the σ-algebra with $\mu(N) = 0$ such that the proposition holds on the complement of N. In this definition, we can say that the proposition holds for every element of the set we are analyzing, except in a subset of measure zero.

A measure λ is *absolutely continuous to the measure μ* in the sense that, if E is in the σ-algebra and $\mu(E) = 0$, then $\lambda(E) = 0$.

As an example of absolute continuity between two measures, let X be the interval $[0, 1]$, and $\mathcal{B}$ the borelian σ-algebra on X. Define μ as the Lebesgue measure on X and let λ assign twice the length of each subset Y of X. Note that λ is absolutely continuous with respect to μ.

Now, let X and μ be defined as in the example above and let ν assign to each subset Y of X, the number of points from the set $\{0.1, ..., 0.9\}$ that are contained in Y. Note that ν is not absolutely continuous with respect to μ, because ν assigns non-zero measure to zero-length sets such as $\mathbb{Q} \cap [0, 1]$.

Once the concepts above are defined and exemplified, we enunciate:

Theorem 4.3.5 (Radon-Nikodym)**.** *Let μ and ν be σ-finite measures on the σ-algebra $\mathcal{A}$ and ν is absolutely continuous with respect to μ. Then there is a function $f \geq 0$ such that*

$$\nu(E) = \int_E f d\mu, \qquad E \in \mathcal{A}.$$

Moreover, $f = \frac{d\nu}{d\mu}$ is called the Radon-Nikodym derivative and it is uniquely determined μ-almost everywhere.

To develop an intuition as to what this theorem says, we can think in terms of probability measure. Let's set $P(A) = \int_A f(x)dx$. With the Radon-Nikodym theorem, we can represent the probability of the set A, $P(A)$, as the density function $f(x)$. The Radon-Nikodym derivative of $P(A)$ is then the density function $f(x)$.

Even though this example can be useful to develop an intuition into the Radon-Nikodym

derivative, we should keep in mind that this theorem is general and applies to arbitrary measures, beyond the scope of probability or Lebesgue measures. It is also valid for arbitrary spaces beyond the Euclidean one.

Now that we have presented the essential results from measure theory that were necessary to build modern probability in a general and abstract context, the next section will present the evolution of the works of mathematicians that tried to connect probability with measure theory or build the theory of probability in a way that would overcome the limitations of the classical approach.

4.4 The search for the axioms and early connections between probability and measure theory.

In this section we expose the evolution of the ideas in probability from its association to measure theory up to the preliminary foundation required for the construction of the axioms. We will begin the exposition by the association of probability and measure as given by Hausdorff and the call for axiomatization by Hilbert. After that, we will present an essential contribution made by Borel's work on denumerable probability, where he introduced countable additivity to probability, introduced the result of the strong law of large numbers and connected binary experiments, like heads and tails, to an uncountable set. Finally, we will present the first attempts at axiomatization and the evolution of probability towards a more abstract context.

4.4.1 The connection of measure and probability and the call for the axioms

The association of probability and measure theory was well established with the work of Felix Hausdorff. Although some association between the two had been previously explored by other authors, in Hausdorff's work, he takes probability as an application of measure theory and gives a rigorous treatment to Poincaré's intuition that probability 0 doesn't necessarily mean impossibility and asserted that many *"theorems on the measure of point sets take on a more familiar appearance when expressed in the language of probability calculus"* [67] (p. 35).

Hausdorff stated that the measure normalized is defined to be a probability. Today we take the opposite approach, that is, probability is defined formally as a measure. Hausdorff's book was considered the standard reference for set theory, and we will consider the connection between

probability and measure theory established in his work of 1914.

At the beginning of the 20^{th} century, classical probability was showing its limits, and mathematicians were searching for a rigorous definition that would formally define terms such as event, trial, randomness and even probability itself. Poincaré says: "*On ne peut guère donner une définition satisfaisante de la Probabilité. On dit ordinairement : la probabilité d'un événement est le rapport du nombre des cas favourables à cet événement au nombre total des cas possibles* [51] (p. 24)".

Hilbert's well known list of open problems in mathematics, published in the International Congress of Mathematicians in Paris in 1900, called for an axiomatization of those parts of physics in which mathematics played an important role, with a special attention to probability and mechanics. Hilbert was concerned about the foundations of statistical mechanics. He was searching for a firm mathematical basis for the determination of average values, that could be found using probability distributions for the quantity considered [67]. In a survey on the works of history from measure to probability, Bingham points to Hilbert's description of probability as a physical science in his call for the axioms as evidence as to the unsatisfactory state of probability. In his own words:

Hilbert's description of probability as a physical science, which one can hardly imagine being made today, is striking, and presumably reflects both the progress in statistical mechanics by Maxwell, Boltzmann and Gibbs and the unsatisfactory state of probability theory at that time judged as mathematics [9] (p. 146).

4.4.2 Borel's denumerable probability

Borel made substantial contribution to probability in his 1909 paper: *Les probabilités dénombrables et leurs applications arithmétiques.* In this paper, he employs the use of countable additivity to probability and also develops an astonishing result: the strong law of large numbers. Borel starts his text saying that there are two categories in probability problems, when the number of possible cases is finite and the continuous probability. He then introduces a new category, the one of the countable sets, which is placed between the finite and the continuous probabilities.

In this same work, Borel takes a number $x \in [0, 1]$ and represents it with binary digits (0's and 1's). Setting, $x = b_1 b_2 \cdots$ and $[0, 1]$ with the Lebesgue measure, Borel shows that $x = b_1 b_2 \cdots$ becomes a random variables with the same distribution used in calculating the probability of the outcome of successive and independent coin tosses. He says that the probability assigned to the event that the n tosses of a coin gives one specific sequence of heads and tails is 2^{-n}. This value is

also the Lebesgue measure of the finite set of intervals whose points x have binary representations with a specified sequence of 0's and 1's in the first n places.

Borel explains the importance that he gives to the countable sets in probability. In his own words: *... cette notion du continu, considéré comme ayant une puissance supérieure à celle du dénombrable, me paraît être une notion purement négative, la puissance des ensembles dénombrables étant la seule qui nous soit connue d'une manière positive, la seule qui intervienne effectivement dans nos raisonnements. Il est clair, en effet, que l'ensemble des éléments analytiques susceptibles d'être réellement définis et considérés ne peut être qu'un ensemble dénombrable; je crois que ce point de vue s'imposera chaque jour d'avantage aux mathématiciens et que le continu n'aura été qu'un instrument transitoire, dont l'utilité actuelle n'est pas négligeable (...), mais qui devra être regardé seulement comme un moyen d'étudier les ensembles dénombrables, lesquels constituent la seule réalité que nous puissions atteindre* [13] (p. 247-248).

Three possible cases of denumerable probabilities are distinguished:

(1) A limited number of possible outcomes in each try, but with a countably infinite number of tries;

(2) Countably infinitely many possible cases in each try, but the number of tries is finite;

(3) The possible cases and the number of tries are both countably infinite.

Borel mentions that he starts by the first case (countable infinite many tries of a finite number of possible outcomes) and begins to present many probability problems. We explain the first three problems, which are the relevant ones for the proof of Borel's strong law of large numbers.

- **Problem 1:** What's the probability that the favourable cases never happen?

Borel denotes p_n the probability of success in the n^{th} trial, and A_0 the probability of the event that a favourable case will never occur, where A_0 is given by: $A_0 = (1-p_1)(1-p_2)\cdots(1-p_n)\cdots$. He excludes any case in which $p_n = 1$ and then concludes that if

$$\sum_{n=1}^{\infty} p_n \tag{3}$$

is convergent, then $0 < A_0 < 1$. In case of divergence of the series (3), $A_0 = \lim_{n\to\infty} \prod_{i=1}^{n}(1 - p_i) = 0$, so A_0 goes to zero as n grows. In this case, Borel takes some caution in his explanation,

recalling that probability zero doesn't necessarily mean impossibility. He recalls his paper of 1905 [12] where he explains that the probability of choosing a rational number at random is zero, but it doesn't mean that there are no rational numbers. Having noted this, Borel concludes that in the case of divergence, $A_0 = 0$, but it only means that the probability that no favorable case will occur goes to zero when the number of trials increases indefinitely.

- **Problem 2:** What's the probability that the favourable cases happen exactly k times?

Borel denotes this probability A_k and starts analyzing the case where $k = 1$.

If the favourable case happens in the first trial we have: $\omega_1 = p_1(1-p_2)(1-p_3)\cdots(1-p_n)\cdots$. If the series (3) is convergent, then $\omega_1 = \frac{p_1}{1-p_1}A_0$. If the series is divergent, $\omega_1 = 0$.

He then presents the case of the favourable case happening in the n^{th} trial as $\omega_n = \frac{p_n}{1-p_n}A_0$.

A_1 will be the sum of all the ω_n and we get: $A_1 = A_0\left(\frac{p_1}{1-p_1} + \frac{p_2}{1-p_2} + \cdots + \frac{p_n}{1-p_n} + \cdots\right)$. The series inside the parenthesis is convergent, and Borel sets: $u_n = \frac{p_n}{1-p_n} = \frac{p_n}{q_n}$ and $A_1 = A_0\sum_{i=1}^{\infty} u_n$.

In case of divergence of the series (3), we also have divergence in the sum of the u_n's and $A_0 = 0$. In this case, A_1 is indeterminate, of the form $0 \cdot \infty$. If we see that A_1 is the sum of the ω_n's, and that each ω_n is zero in the divergent case, we have that A_1 is a countable sum of zeros, so it should be zero. Borel doesn't feel comfortable using this fact, saying that even if the ω's are all zero, there are infinitely many of them, so we can't conclude without caution that their sum is zero, if we keep in mind that zero probability doesn't necessarily mean impossibility. So he develops an argument and finally concludes the result that $A_1 = 0$ in the divergent case.

After this, he gives the result that $A_k = A_0\sum u_{n_1}u_{n_2}\ldots u_{n_k}$ if (3) is convergent and $A_k = 0$ if the series is divergent.

- **Problem 3:** What's the probability that the favourable cases happen an infinite number of times?

Borel starts by denoting A_∞, the probability of favourable cases happening an infinite number of times. He then considers the case where (3) is convergent and evaluates the sum: $S = A_0 + A_1 + \cdots + A_k + \cdots$. He says that by the previous results on the A_k we can write: $S = A_0(1+u_1)(1+u_2)\cdots(1+u_k)\cdots$. Now using the fact that $A_0 = (1-p_1)(1-p_2)\cdots(1-p_n)\cdots$ and $u_n = \frac{p_n}{1-p_n} = \frac{p_n}{q_n}$, we have that $1 = u_n = \frac{1}{1-p_n}$. Taking the product, we get $\prod(1+u_n) = \prod\frac{1}{1-p_n} = \frac{1}{A_0}$

and we can finally write: $S = A_0 \frac{1}{A_0} = 1$. To conclude, A_∞ is exactly the complement of S, so $A_\infty = 1 - S = 0$.

In the case of divergence of the series (3), each $A_k = 0$, so $S = 0$ and $A_\infty = 1$, however Borel again develops an argument to show this result because he wasn't comfortable with summing zeros a countable infinite number of times.

With the three problems presented here, we have demonstrated the following result:

Theorem 4.4.1 (Borel 0-1 law). *Let's take a countable infinite sequence of independent binary events, where p_n is the probability of a favorable case occurring in the n^{th} trial.*
If $\sum_{n=1}^{\infty} p_n < \infty$, then $A_\infty = 0$.
If $\sum_{n=1}^{\infty} p_n = \infty$, then $A_\infty = 1$.

A few years later, Cantelli remarked that the hypothesis of independence of the Borel 0-1 law could be relaxed and this new result is known as the Borel-Cantelli lemma.

Borel applies his 0-1 law with the dyadic expansion of a real number x chosen at random in $[0, 1]$ and he developed an astonishing result, the strong law of large numbers, which we will now present.

Any $x \in [0, 1]$ can be written as: $x = \cdot b_1 b_2 \dots b_n \dots = \sum_{n=1}^{\infty} \frac{b_n}{2^n}$, where each b_n is either 0 or 1. When the sequence (b_n) is generated, or equivalently x is chosen, each digit b_n has probability $1/2$ of being 0 or 1 and the digits $n = 1, 2, \dots$ are independent trials [4].

Borel adopted 0 as the favourable case and stated that if we take $2n$ trials, the probability that the number of favourable cases will be between

$$n - \lambda\sqrt{n} \quad \text{and} \quad n + \lambda\sqrt{n}$$

is given by

$$\Theta(\lambda) = \frac{2}{\sqrt{\pi}} \int_0^{\lambda} e^{-\lambda^2} d\lambda$$

and this probability converges to 1 as λ increases.

Borel takes a sequence (λ_n), with $\lambda_n = \log n$, so (λ_n) is an increasing sequence such that $\lim_{n\to\infty} \frac{\lambda_n}{\sqrt{n}} = 0$.

The first $2n$ trials give a favourable result if the number of times that 0 appears will be between

$$n - \lambda_n\sqrt{n} \quad \text{and} \quad n + \lambda_n\sqrt{n}.$$

The probability p_n of the favourable case is:

$$p_n = \Theta(\lambda_n) = \frac{2}{\sqrt{\pi}} \int_0^{\lambda_n} e^{-\lambda^2} d\lambda.$$

Now Borel sets $q_n = 1 - p_n$. The sum of q_n is convergent, so the probability of having non-favourable cases infinitely many times is 0. He concluded that, after a certain value n, the probability of constantly being in the favourable case is 1. The ratio between the number of 0's and the number of 1's will be between:

$$\frac{n - \lambda_n\sqrt{n}}{n + \lambda_n\sqrt{n}} \quad \text{and} \quad \frac{n + \lambda_n\sqrt{n}}{n - \lambda_n\sqrt{n}}, \quad \text{or equivalently between} \quad \frac{1 - \lambda_n/\sqrt{n}}{1 + \lambda_n/\sqrt{n}} \quad \text{and} \quad \frac{1 + \lambda_n/\sqrt{n}}{1 - \lambda_n/\sqrt{n}}.$$

One big flaw of Borel's proof here is that he assumes the convergence of the p_n's according to the Central Limit Theorem, however the classic version of that theorem considers independent and identically distributed random variables, which is not the case because λ_n is not fixed [4]. Also, as pointed out in [67], the convergence of the $\sum q_n = \sum(1 - p_n)$ is not guaranteed by the convergence of the series of the $\Theta(\lambda_n)$.

Even though the proof of Borel's strong law is not perfect, the authors that came after him were able to fix it, as will be seen in the last section of this chapter. But a question that arises and needs to be raised at this point is: What is the innovation of this result? What is the difference between the weak and the strong law of large numbers?

To answer these questions, let's denote by $\nu_{2n}(x)$ the number of 0's in the first $2n$ trials of a binary experiment. While the weak law, in today's version, states that $\lim_{n\to\infty} P\left(\left|\frac{\nu_{2n}(x)}{2n} - \frac{1}{2}\right| > \varepsilon\right) = 0$, the strong law states that $P\left(\lim_{n\to\infty} \frac{\nu_{2n}(x)}{2n} = \frac{1}{2}\right) = 1$.

This means that the weak law states a probable proximity, but doesn't guarantee a convergence for the relative frequency. That is, with a sufficiently large sample, there will be a very high probability that the average of the observations will be within an arbitrarily small interval around the expected value, but it is still possible that $|\overline{X}_n - \mu| > \varepsilon$ happens an infinite number of times, although at infrequent intervals.

The strong law doesn't leave room for this possibility to happen, because it says that there is a probability 1 that the limit always applies, that is, for any $\varepsilon > 0$ the inequality $|\overline{X}_n - \mu| < \varepsilon$ holds for all n large enough.

4.4.3 The first attempts at axiomatization

An early attempt at axiomatization came from Laemmel in 1904. He had worked on the independent case and discussed the rules of total and compound probability as axioms, but didn't give any explanation of the concept of independence [56].

Ugo Broggi's dissertation under Hilbert's direction in 1907, proposed two axioms: i) the certain event has probability 1, and ii) the rule of total probability. After these axioms, he defined probability as a ratio of the number of cases for a discrete set, and the ratio of the Lebesgue measures in the geometric setting. To Broggi, total probability implied countable additivity, which would later be contested by Steinhaus. This last one mentions the generalization of Lebesgue's measure for all the subsets E of the interval $[0, 1]$ given by Banach, that shows the existence of a function $\mu(E)$ which is finite additive but not countably additive[63].

From 1918 to 1920, Daniell developed the integral of a linear operator on some class of continuous real-valued functions on an abstract set E. Applying Lebesgue's methods in this general setting, Daniell extended the linear operator to the class of summable functions. Using ideas from Fréchet, Daniell also gave examples in infinite-dimensional spaces and used his theory of integration to construct a theory of Brownian motion.

In November 1919, Wiener submitted an article where he laid out a general method for setting up Daniell's integral when the underlying space E is a function space. Daniell was aware of the importance of Brownian motion and of its model in physics made by Einstein. He then followed with a series of articles where he used Daniell's integral to formalize the notion of Brownian motion on a finite time interval.

In 1923, Antoni Lomnicki published an article where he proposed that probability should be faced relative to a density ϕ on a set $\mathcal{M}$ in $\mathbb{R}^n$. He had used two ideas from Carathéodory: the first one was that of a p-dimensional measure and the second one was that of defining the integral of a function on a set as the measure of the region between the set and the function's graph. To Lomnicki, the probability of a subset $m \subset \mathcal{M}$ is the ratio of the measure of two regions: that one between m and ϕ's graph and that between $\mathcal{M}$ and this graph. Together with Ulam, Lomnicki was the first to take probability outside the geometric context and lead it to abstract spaces. Ulam, at the 1932 International Congress of Mathematicians in Zurich, announced that Lomnicki had shown that product measures[9] can be constructed in abstract spaces. Ulam asserted that their

[9] If $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are measure spaces, then there is a measure π, called the *product measure*, defined on the

probability measure satisfies the same conditions on the product space of a countable sequence of spaces. Their idea can be put in today's language as: m is a probability measure on a σ-algebra that is complete[10], that is, includes all null sets, and contains all singletons. Ulam and Lomnicki's axioms were published in 1934 citing Kolmogorov's Grundbegriffe as an authority to their work.

Von Mises was a mathematician concerned with applied studies who aimed to create a statistical physics freed from mechanical assumptions. In his point of view, classical mechanics cannot serve as a foundation for statistical physics and a genuine probabilistic behaviour is not compatible with a mechanical description. After this point of view, he made significant contributions in formulating a system for statistical physics based on the use of Markov chains.

He has his name associated with the frequentist approach in probability, and was pointed by some authors as "*a crank semimathematical theory serving as a warning of the state of probability before the measure theoretic revolution*" [67] (p. 180). What is striking in this story is that Kolmogorov himself based the application of probability on von Mises' ideas, as he explains in a foot-note of the Grundbegriffe: *The reader who is interested in the purely mathematical development of the theory only, need not read this section, since the work following it is based only upon the axioms in §1 and makes no use of the present discussion. Here we limit ourselves to a simple explanation of how the axioms of the theory of probability arose and disregard the deep philosophical dissertations on the concept of probability in the experimental world. In establishing the premises necessary for the applicability of the theory of probability to the world of actual events, the author has used, in large measure, the work of R. v. Mises.* [39] (p. 3).

Von Mises published a work in 1919 concerned with the foundations of probability, where he proposed a foundational system. This system was based on a sample space of possible results, each represented by a number, with an experiment that is repeated indefinitely. The resulting sequence of numbers is called a *collective* if: i) the limits of relative frequencies in that sequence exist and ii) these limits remain the same in subsequences formed of original sequence. From the definition of collectives, probability is defined as the limit of relative frequency, with the second item giving us the postulate of randomness.

As one of the founders of logical empiricism, von Mises considered mathematical infinity an idealization that could not claim empirical reality directly, but only as a useful tool. One of the most

subsets of $Z = \mathcal{A} \times \mathcal{B}$ such that $\pi(A \times B) = \mu(A)\nu(B)$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$ [53].

[10] A measure space $(X, \mathcal{M}, \mu)$ is said to be *complete* provided $\mathcal{M}$ contains all subsets of sets of measure zero, that is, if E belongs to $\mathcal{M}$, and $\mu(E) = 0$, then every subset of E also belongs to $\mathcal{M}$ [53].

important critiques of von Mise's collectives came from Jean Ville [66]. Regarding the randomness postulate, one may ask if a property appears as truly randomly distributed in a population or if a different frequency of the outcomes could be obtained by a more informed way of sampling. This critique can be associated with the result of a sequence of 0's and 1's. If the limit of a sequence of relative frequencies is neither 0 nor 1, then it can be different from a subsequence composed by only 0's or only 1's.

Ville's strongest objection to von Mise's collectives is that this theory is not compatible with countable additivity. His argument relys on the limit theorems, where convergence occurs infinitely often. Ville stated that there is a collective such that the frequency of 1's in the sequence is always greater than or equal to p. In this same work, Ville creates the important concept in probability of Martingale.

In 1922, a paper written by the Soviet mathematician Eugen Slutsky provided a new approach to the development of probability theory, which he devised while trying to answer Hilbert's 6^{th} problem. In his attempt to make probability purely mathematical, he removed the word probability and the idea of equally likely cases from the theory. This was the first time that probability theory did not depend on equally likely cases. According to Slutsky, to develop this theory, instead of bringing up equally likely cases, one should start by just assuming that numbers are assigned to cases, and when a case that has been assigned with the number α is divided in sub-cases, the sum of the numbers of the sub-cases should add to α. It is not required that each case has the same number. Slutsky proposed something very general that he called "valence", with three possible interpretations: i) classical probability, based on equally likely cases, ii) finite empirical sequences and iii) limits of relative frequencies. So it can be said that probability would be one possible interpretation for Slutsky's valences. To Slutsky, probability could not be reduced to limiting frequency, as the latter has very limiting properties to the former.

In the year following the publishing of Slutsky's paper, Steinhaus [63] proposed a set of axioms to Borel's theory of denumerable probability. He defined:

- A as the set of all possible infinite sequences of heads and tails (H and T);
- $E, E', \ldots$ as subsets of A;
- E_n as subsets of A with first n elements in common, $n = 0, 1, 2, \ldots$ or ∞;
- $\mathfrak{M}$ as a class of all subsets of A and

- $\mathfrak{K}$ as a class of certain subsets of E, that is, the class $\mathfrak{K}$ is part of $\mathfrak{M}$.

Then he sets μ to be a set function defined for all $E \in \mathfrak{K}$ such that:

(1) $\mu(E) \geq 0$ for all $E \in \mathfrak{K}$;

(2) (a) $\mu(A) = 1$;

(b) $E_n \in \mathfrak{K}$;

(c) If two sets E_n and $E_n^{'}$ only differ in the i^{th} element, $(i \geq n)$, then $\mu(E_n) = \mu(E_n^{'})$;

(3) $\mathfrak{K}$ is closed under finite and countable unions of disjoint elements, and μ is finitely and countably additive.

(4) If $E_2 \subset E_1$, and E_1 and E_2 are in $\mathfrak{K}$, then $E_1 \backslash E_2$ is in $\mathfrak{K}$.

(5) If E is in $\mathfrak{K}$ and $\mu(E) = 0$, then any subset of E is in $\mathfrak{K}$.

Steinhaus concluded that the theory of probability for an infinite sequence of binary trials is isomorphic to the theory of Lebesgue measure. Although Steinhaus considered only binary trials, his reference to Borel's more general concept of denumerable probability opened paths to further generalizations [56].

Kolmogorov himself made significant contributions to probability theory before publishing his axioms. In 1925's article with Khinchin [38] Kolmogorov proved the convergence, with probability 1, of a series of random variables and also gave the sufficient and necessary conditions for that convergence. In 1928, Kolmogorov wrote an article [37] where he proved what he called the *generalized law of large numbers*, which is a version of the strong law for independent random variables. Kolmogorov's article of 1929 [35] defines several probability ideas using measure theory. He expresses his concerns with the possibility of constructing a very general and purely mathematical theory to solve probability problems. In this article he considered a set A endowed with a measure M, (A, M) is a metric space, and some subsets $E \subset A$. Then he defined three axioms for his measure: i) $M(E) \geq 0$; ii) if $E_1 \cap E_2 = \emptyset$, then $M(E_1 \cup E_2) = M(E_1) + M(E_2)$, and; iii) $M(A) = 1$. From these axioms, he showed some standard results in probability but what calls attention in this work is the use of countable additivity. He defined a normal measure as one where countable additivity holds. This concept necessary to justify arguments involving the

convergence of random variables. In his work of 1931 [36], on continuous time stochastic process, Kolmogorov freely uses countable additivity and also Fréchet's framework for abstract sets.

Cantelli constructed a theory with no appeal to empirical notions, such as possibility, event, probability or independence. His theory started with an abstract set of points with positive and finite measure. We can enumerate his definition from [56]:

(1) $m(E)$ is the area of a subset E;

(2) $m(E_1 \cap E_2) = m(E_1) + m(E_2)$ when E_1 and E_2 are disjoint;

(3) $0 \leq m(E_1E_2)/m(E_i) \leq 1$, for $i = 1, 2$.

(4) E_1 and E_2 are called *multipliable* when $m(E_1E_2) = m(E_1)m(E_2)$.

Even though Cantelli's work was general and abstract, Kolmogorov's works of 1929 and 1931 had already gone beyond Cantelli's contributions in abstraction and mathematical clarity. However, it's important to note that Cantelli had developed, independently of Kolmogorov, the combination of a frequentist interpretation of probability with an abstract axiomatization that incorporated classical rules of total and compound probability [56].

4.4.4 The proofs of the strong law of large numbers

Borel's strong law of large number was a quite surprising result: the measure of binary decimals with a limiting frequency of 1's different from 1/2, is zero. Following, Borel's result, many mathematicians started to work on the strong law of large numbers to improve its results. Faber constructed a continuous function f where the set of points x where f doesn't have a derivative has Lebesgue measure 0. Letting $n(1)$ and $n(0)$ denote the numbers of 1's and 0's, respectively, in the first n binary digits of x, if $\liminf(n(1)/n(0)) < 1 - \varepsilon$ or $\limsup(n(1)/n(0)) > 1 + \varepsilon$, then there is no derivative. It follows that the set of x for which $\lim(n(1)/n(0)) = 1$ has measure 1 [56].

Hausdorff also proves Borel's strong law of large numbers. Putting $n(1)$ as above, Hausdorff shows that $n(1)/n \to 1/2$ as $n \to \infty$, except on a set of measure 0. He then studies the asymptotic behavior of the oscillation of frequency. Hausdorff found limits of $\pm \log n\sqrt{n}$ for the deviation of the number $n(1)$ from the average $n/2$ [56].

Hardy and Littlewood [31] (p. 185) have shown that, with $n(1)$ as above, $\frac{|n(1)-n/2|}{\sqrt{n \log n}} \to 1$ as $n \to \infty$, except for on a set of measure 0. They point out that $\sqrt{n \log n}$ is an upper bound for the deviation of the frequency $|n(1) - n/2|$ and that $\sqrt{n}$ can be improved as a bound, because $\liminf |n(1) - n/2| > \sqrt{n}$ (p. 187).

In 1923, Khintchin improved Hardy and Littlewood's upper bound to $\sqrt{n \log \log n}$. This result is known as the law of iterated logarithm and one year later he was able to show that this bound cannot be improved. Khintchin considered a simple event with probability of success p and has shown that there is a function $\chi(n)$ such that for any ε and δ, there is a natural number n_0 such that, with a probability greater than $1 - \delta$ we have, for all $n > n_0$, the inequality: $1 - \varepsilon < \left| \frac{n(1)-n/2}{\chi(n)} \right| < 1 + \varepsilon$. The solution gives with $q = 1 - p$ the asymptotic expression: $\sqrt{2pqn \log \log n}$.

Maistrov [46] presents Khintchin's idea geometrically. In Figure (4.6), the values of n are placed on the x-axis and the values of $n(1) - n/2$ on the y-axis. Then two straight lines, $y = \varepsilon n$ and $y = -\varepsilon n$, are drawn. By the Borel-Cantelli lemma, for n large enough, the value $n(1) - n/2$ will almost certainly stay between the lines $y = \varepsilon n$ and $y = -\varepsilon n$. What Khintchin had accomplished to do was to find that for any ε and n large enough, the quantity $n(1) - n/2$ will stay with near certainly within the curves:

- $y = (1 + \varepsilon)(2npq \log \log n)^{1/2}$ (l)
- $y = -(1 + \varepsilon)(2npq \log \log n)^{1/2}$ (l')

and outside the curves

- $y = (1 - \varepsilon)(2npq \log \log n)^{1/2}$ (ll)
- $y = -(1 - \varepsilon)(2npq \log \log n)^{1/2}$ (ll')

infinitely often.

Khintchin was able to show that if the probability of occurence of the event A in each of the n independent trials is equal to p, then the number $n(1)$ of occurrences of the event A in n trials satisfies:

$$P\left(\limsup_{n\to\infty} \frac{n(1) - n/2}{(2npq \log \log n)^{1/2}} = 1\right) = 1.$$

In 1928, Khintchin showed that if a sequence of random variables were independent and identically distributed, the existence of the expectation was a necessary and sufficient condition to

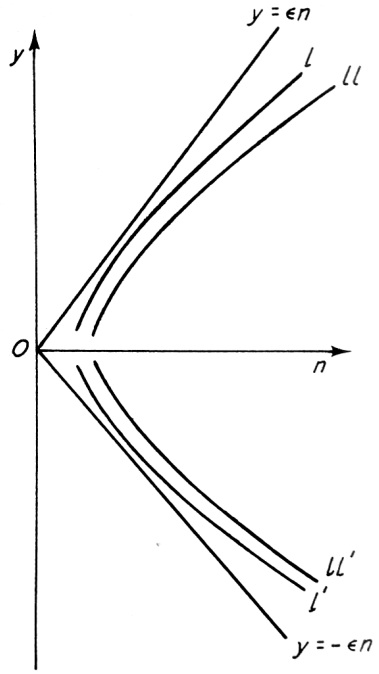


Figure 4.6: Khintchin's bounds - [46] (p. 260).

apply the weak law of large numbers. Kolmogorov discovered the conditions to be imposed on a sequence of random variables in order for the strong law of large numbers to hold, which in the case of independent and identically distributed random variables, is the existence of expectation.

In all of these investigations, the analogy with metric theory of functions played a significant role, and Kolmogorov started to engage in the logical formulation of these ideas which ended in the formulations of the axioms of probability that we describe in the next chapter.

Chapter 5

Kolmogorov's foundation of probability

5.1 Introduction

Andrei Nikolaevich Kolmogorov was born in Tambov, Russia, in 1903, and died in Moscow, in 1987. Kolmogorov had wide-ranging intellectual interests, including Russian history and Aleksandr Pushkin's poetry. Kolmogorov entered Moscow University in 1920 to study mathematics. Following his graduation in 1925 and his doctorate four years later, he became a professor at Moscow University's Institute of Mathematics and Mechanics in 1931. Kolmogorov taught mathematically gifted children for many years and served as the director for almost seventy advanced research students, many of whom became significant mathematicians in their own right. He is considered one of the 20^{th} century's greatest mathematicians, with a rarely found level of creativity and versatility. Besides probability, Kolmogorov also made contributions to many other fields, such as algorithmic information theory, the theory of turbulent flow, dynamical systems, ergodic theory, Fourier series, and intuitionistic logic [26].

In his book *Foundations of the Theory of Probability*, Kolmogorov was able to identify the contributions from many authors, including himself, and summarize those findings in such a powerful way that the work of those who came before him became overshadowed by the synthesis that he did. Kolmogorov developed the subject in a fully abstract way, beyond Euclidean spaces, and formalized terms that had previously been only loosely defined (such as event, random variable and

even probability). This ability to capture the most essential ideas and create a set of abstract axioms put an end to classical probability and started the era of modern probability when this discipline became an autonomous branch of mathematics.

In the preface, Kolmogorov says that the purpose of his book is to give an axiomatic foundation for probability. As he said: "*the author set himself the task of putting in their natural place, among the general notions of modern mathematics, the basic concepts of probability theory*" (p. v). Besides the historical exposition of the evolution of probability before Kolmogorov's book presented in chapter four, his own words establish the purpose of his work as one of synthesis. "*While a conception of probability theory based on the above general viewpoints has been current for some time among certain mathematicians, there was lacking a complete exposition of the whole system, free of extraneous complications*" (p. v). Nonetheless, his book also makes some advances and innovations to science. Besides the axioms, Kolmogorov also exposes other original contributions such as probability distributions in infinite-dimensional spaces (Chapter III, §4), which provided a framework for the theory of stochastic processes; differentiation and integration of mathematical expectations with respect to a parameter (Chapter IV, §5); a general treatment of conditional probabilities and expectations (Chapter V), built on Radon-Nikodyn's theorem. As Kolmogorov mentions: "*It should be emphasized that these new problems arose, of necessity, from some perfectly concrete physical problems*" (p. v).

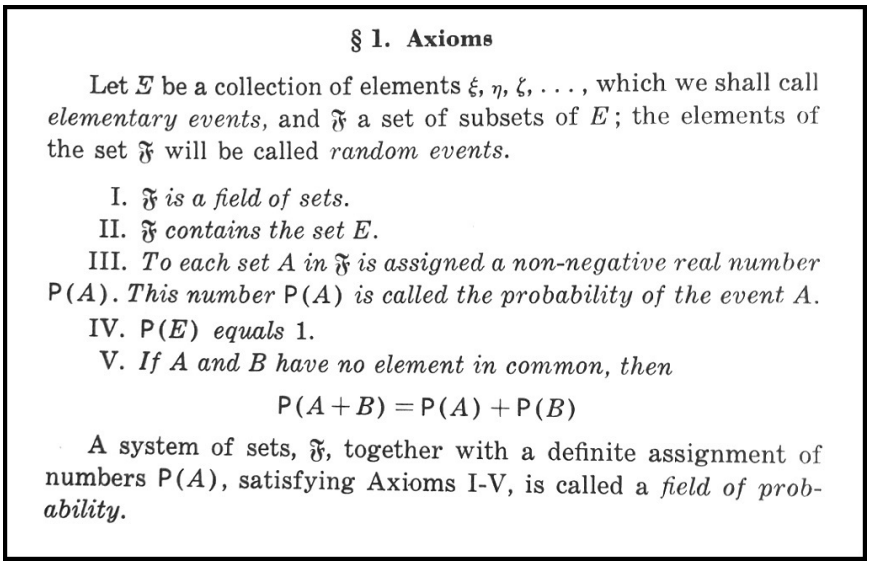
Kolmogorov's book, constructs the axiomatization in two chapters. In the first one, he presents five axioms considering a finite sample space. The main contribution there is the set of axioms that formalized and generalized the classical definition in finite spaces. The second chapter adds another innovation to the definition, because it reaches its full generality when Kolmogorov introduces axiom VI and takes probability to infinite spaces. After this introduction, in the next section, we present the definitions of probability from Kolmogorov's book and demonstrate in more detail some theorems that he gave an abbreviated proof. The third section presents the concepts of probability functions, random variables and conditional probability according to Kolmogorov's developments, and we present conditional mathematical expectation following modern textbooks. In the last section, we illustrate how Kolmogorov's work has established the grounds for probability theory free of ambiguities. In order to do so, we present an example that leads to a paradox in classical probability which is resolved by Kolmogorov's new approach using conditional probability.

5.2 Kolmogorov's axioms of probability

5.2.1 Elementary theory of probability

Kolmogorov's definition of probability is given in the first two chapters of his book. The first one is restricted to what he called *elementary theory of probability*, which is set up in finite sample spaces. In the second chapter he introduces another axiom that enables us to work with infinite probability spaces.

Figure (5.1)[1] presents Kolmogorov's axioms I through V from the first chapter of his book.



§ 1. Axioms

Let $\mathcal{E}$ be a collection of elements $\xi, \eta, \zeta, \ldots$, which we shall call *elementary events*, and $\mathfrak{F}$ a set of subsets of E; the elements of the set $\mathfrak{F}$ will be called *random events*.

I. *$\mathfrak{F}$ is a field of sets.*
II. *$\mathfrak{F}$ contains the set E.*
III. *To each set A in $\mathfrak{F}$ is assigned a non-negative real number $\mathsf{P}(A)$. This number $\mathsf{P}(A)$ is called the probability of the event A.*
IV. *$\mathsf{P}(E)$ equals 1.*
V. *If A and B have no element in common, then*

$$\mathsf{P}(A+B) = \mathsf{P}(A) + \mathsf{P}(B)$$

A system of sets, $\mathfrak{F}$, together with a definite assignment of numbers $\mathsf{P}(A)$, satisfying Axioms I-V, is called a *field of probability*.

Figure 5.1: Kolmogorov's axioms I to V - [39] (p. 2).

After presenting the axioms, Kolmogorov presents a brief discussion on how to construct fields of probability. He takes a finite set $E = \{\xi_1, \xi_2, \ldots, \xi_k\}$ and a set of non-negative numbers $\{p_1, p_2, \ldots p_k\}$ with the sum $p_1 + p_2 + \ldots + p_k = 1$. $\mathfrak{F}$ is the set of all subsets of E and $P(\xi_{i_1}, \xi_{i_2}, \ldots, \xi_{i_k}) = p_{i_1} + p_{i_2} + \ldots + p_{i_k}$. The p_i's are called probabilities of the elementary events $\{\xi_i\}$'s.

Along with the definition of probability, in the rest of the 1^{st} chapter, Kolmogorov presents some corollaries from the axioms, the definition of conditional probabilities, independence, Markov chains and the theorem of Bayes. It is remarkable that he advises the reader who is interested in purely mathematical development to skip §2, where he indicates a frequentist interpretation of

[1] Kolmogorov denoted E as the whole sample space. With the exception of this figure, which is a screen-shot of Kolmogorov's book, we will denote this space as Ω which is the most common notation in modern texts.

probability without getting into the details but suggesting von Mises as a reference. He also mentions that an impossible event - an empty set - has probability 0, but the converse doesn't hold: there are sets A such that $P(A) = 0 \not\Rightarrow A$ is an impossible event. When $P(A) = 0$, the event A still can happen in a long series, but not very often.

5.2.2 Infinite probability fields

Everything that was said in the previous subsection concerned finite probability spaces. In his second chapter, Kolmogorov introduces axiom VI as shown in Figure (5.2),[2] which is the missing ingredient that enables one to work with infinite probability fields. Note that the first five axioms are related to an algebra of sets to define probability. Now the axiom VI establishes the continuity of the probability. The concepts of a σ-algebra and countable additivity from measure theory are crucial for this passage from finite to infinite spaces.

VI. ***For a decreasing sequence of events***

$$A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots \qquad (1)$$

of $\mathfrak{F}$***, for which***

$$\mathfrak{D}_n A_n = 0 \quad , \qquad (2)$$

the following equation holds:

$$\lim P(A_n) = 0 . \qquad n \to \infty \qquad (3)$$

Figure 5.2: Kolmogorov's axiom VI - [39] (p. 14).

This axiom states that probability is a continuous set function at $\emptyset$, that is, for any decreasing sequence of sets $A_1 \supset A_2 \supset \ldots$ of $\mathfrak{F}$, we have that $\lim_{n\to\infty} P(A_n) = 0$. Subsequently, Kolmogorov presents the *Generalized Addition Theorem* where, from the finite additivity and continuity at $\emptyset$ (axiom V and VI), he shows that probability is countably additive[3]. Note that this idea of countable additivity in measurable sets comes from Borel, as we have shown in chapter four.

Theorem 5.2.1 (Generalized Addition Theorem). *If* $A_1, A_2, \ldots, A_n, \ldots$ *and* A[4] *belong to* $\mathfrak{F}$*, then from* $A = \cup_n A_n$*, follows the equation* $P(A) = \sum_n P(A_n)$.

[2] In Kolmogorov's notation, $\mathfrak{D}_n A_n = A_1 \cap A_2 \cap \ldots \cap A_n$, and $0 = \emptyset$.

[3] Kolmogorov uses the expression *completely additive set function on* $\mathfrak{F}$ as a synonym of countably additive.

[4] $A_1, A_2, \ldots, A_n, \ldots$ and A are pairwise disjoint. Kolmogorov doens't mention it when he states the theorem, but he uses this fact in the proof.

Proof. Let's set $R_n = \cup_{m>n} A_m$. As (A_n) is an infinite sequence of disjoint sets, (R_n) is a decreasing sequence of sets such that $\mathfrak{D}_n(R_n) = \emptyset$. Therefore, by axiom VI, $\lim_{n\to\infty} P(R_n) = 0$.

By axiom V (finite additivity), we can write: $P(A) = P(A_1) + P(A_2) + \ldots + P(A_n) + P(R_n)$. Now, as $\lim_{n\to\infty} P(R_n) = 0$, we have $P(A) = \sum_n P(A_n)$. □

This theorem has shown that the probability $P(A)$ is a countably additive set function on $\mathfrak{F}$. Kolmogorov mentioned without proof that the opposite direction is also true, that is, a countably additive set function is continuous at $\emptyset$. Using a result in [59] (p. 162) as a reference, we will prove this last result and also show that continuity at $\emptyset$ is equivalent to continuity. Our goal here is to show that continuity at $\emptyset$, continuity and countable additivity are equivalent statements in case of probability.

Theorem 5.2.2. *Let P be a countably additive set function defined over the measurable space $(\Omega, \mathfrak{F})$, with $P(\Omega) = 1$. Then P is continuous, which trivially implies that P is continuous at $\emptyset$.*

Proof. First step: under the hypotheses of the theorem, from countable additivity, we will show that P **is continuous from below**, that is: for any increasing sequence of sets $A_1 \subset A_2 \subset \ldots$ of $\mathfrak{F}$, we have $P\left(\cup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} P(A_n)$.
We can decompose $\cup_{n=1}^{\infty} A_n$ into a disjoint union of sets: $\cup_{n=1}^{\infty} A_n = A_1 \cup (A_2 \backslash A_1) \cup (A_3 \backslash A_2) \cup \ldots$, so we have:

$$\begin{aligned} P(\cup_{n=1}^{\infty} A_n) &= P(A_1) + P(A_2 \backslash A_1) + P(A_3 \backslash A_2) + \ldots \\ &= P(A_1) + P(A_2) - P(A_1) + P(A_3) - P(A_2) + \ldots \\ &= \lim_{n\to\infty} P(A_n) \end{aligned}$$

Second step: taking continuity from below, we will show the **continuity from above**, that is, for any decreasing sequence of sets $A_1 \supset A_2 \supset \ldots$ of $\mathfrak{F}$, we have $P\left(\cap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} P(A_n)$. As $\{A_n\}$ is a decreasing sequence of sets, take $n \geq 1$, so $P(A_n) = P(A_1 \backslash (A_1 \backslash A_n)) = P(A_1) - P(A_1 \backslash A_n)$. The sequence $\{A_1 \backslash A_n\}_{n\geq 1}$ is nondecreasing and $\cup_{n=1}^{\infty} (A_1 \backslash A_n) = A_1 \backslash \cap_{n=1}^{\infty} A_n$.

From the first step we get that $\lim_{n\to\infty} P(A_1 \backslash A_n) = P(\cup_{n=1}^{\infty} (A_1 \backslash A_n))$. Now we can set:

$$\begin{aligned} \lim_{n\to\infty} P(A_n) &= P(A_1) - \lim_{n\to\infty} P(A_1 \backslash A_n) \\ &= P(A_1) - P(\cup_{n=1}^{\infty} (A_1 \backslash A_n)) = P(A_1) - P(A_1 \backslash \cap_{n=1}^{\infty} A_n) \\ &= P(A_1) - P(A_1) + P(\cap_{n=1}^{\infty} A_n) = P(\cap_{n=1}^{\infty} A_n) \end{aligned}$$

The continuity at $\emptyset$ trivially follows from the second step. □

After showing that finite additivity and continuity imply countable additivity, Kolmogorov introduces a new definition:

Definition 5.2.1. Let Ω be an arbitrary set, $\mathfrak{F}$ a field of subsets of Ω, containing Ω, and $P(A)$ a non-negative countably additive set function defined on $\mathfrak{F}$; the triple $(\Omega, \mathfrak{F}, P)$ forms a **field of probability**[5].

After defining a field of probability using an algebra of sets $\mathfrak{F}$, Kolmogorov presents a version of the Carathéodory extension theorem from the previous chapter to extend a probability measure from an algebra to the Borel σ-algebra of sets, $B\mathfrak{F}$, generated from the sets of $\mathfrak{F}$:

Theorem 5.2.3 (Extension Theorem)**.** *There is always a unique extension of a non-negative countably additive set function $P(A)$ defined in an algebra $\mathfrak{F}$, to the Borel σ-field, $B\mathfrak{F}$, without losing the properties of non-negativiness and countable additivity.*

We may ask why did Kolmogorov split his definition of probability into two chapters? Why did Kolmogorov start his book with finite probability spaces over algebras and in the second chapter he introduces axiom VI, that defines probability spaces over σ-algebras and allows us to work with infinite spaces. In modern days, we see a σ-algebra as a restriction of an algebra, because the former needs to be closed under countable unions of sets while the latter requires only finite unions. However, Kolmogorov seems to adopt a different point of view. Apparently finite spaces have more empirical appeal and are easier to interpret than infinite ones. "*... the Axiom of Continuity, VI, proved to be independent of Axioms I - V. Since this new axiom is essential for infinite fields of probability only, it is almost impossible to elucidate its empirical meaning [...]. For, in describing any observable random process we can obtain only finite fields of probability. Infinite fields of probability occur only as idealized models of real random processes*" (p. 15).

Following the extension theorem, he makes a remark saying that: the sets of an algebra can be interpreted as observable events but ones from a σ-algebra may not. A σ-algebra is just a

[5]Consider a measure space, $(\Omega, \mathfrak{F}, \mu)$. This space is complete if: for any measure 0 set $A \in \mathfrak{F}$ we have: $C \subset A \Rightarrow C \in \mathfrak{F}$ and $\mu(C) = 0$. The space $([0,1], \mathfrak{B}([0,1]), Leb)$ is not a complete space. $\mathfrak{B}([0,1])$ is smaller than the family of Lebesgue measurable sets. The complete space $([0,1], Leb([0,1]), Leb)$ is a probability space. The Borel σ-algebra is sufficient for all important theorems and completions are mostly an unnecessary complication that results only in loss of tangibility, so won't be used in this thesis. Kolmogorov was conscious that this space was not complete, as he mentioned on page 15 [39].

mathematical structure and its sets are ideal events, without a correspondent in the outside world. But he justifies the use of a σ-algebra by mentioning that the reasoning with a σ-algebra leads to non-contradictory results. "*However if reasoning which utilizes the probabilities of such ideal events leads us to a determination of the probability of an actual event of $\mathfrak{F}$, then, from an empirical point of view also, this determination will automatically fail to be contradictory*" (p.18).

5.3 Definitions in modern probability

In this section we introduce some definitions that were loosely defined in classical probability and formalized in Kolmogorov's book. As it is mentioned in the preface of his *Foundations of the Theory of Probability*: "*While a conception of probability theory based on the above general viewpoints has been current for some time among certain mathematicians, there was lacking a complete exposition of the whole system, free of extraneous complications*" (p. v). Along with the definition of probability that was presented in the previous section, we will introduce the concepts of random variables, mathematical expectation and conditional probabilities according to Kolmogorov's formalization. These definitions will be useful to show how Kolmogorov's axioms and developments set a solid base free of ambiguities to probability as we demonstrate in the following section.

5.3.1 Probability functions and random variables

Kolmogorov starts his chapter of random variables introducing the concept of a partition, which is a function that decomposes our space Ω into disjoint subsets. This definition is important because it prepares the ground for some more advanced results and provides an intuition into measurable functions [1].

Definition 5.3.1. A family $\mathfrak{U}$ of subsets of Ω is a **decomposition** or a **partition** of Ω if its elements are pairwise disjoint and their union is Ω.

Usually a partition is represented as $\mathfrak{U} = \{A_i : i \in I\}$ and I is an arbitrary index set. Another way to represent a partition is by a function u, from Ω to I as $u : \omega \to i$, where i is such that $\omega \in A_i$.

Let's consider two sets of elementary outcomes Ω, Ω' and a function $u : \Omega \to \Omega'$. $u^{-1}[A']$ is the pre-image of A' under u: $u^{-1}[A'] = \{\xi \in \Omega : u(\xi) \in A'\}$. For singletons we will denote: $u^{-1}(a) = u^{-1}[\{a\}] = \{\omega \in E : u(\omega) = a\}$.

To each subset of Ω', we assign the probability of its pre-image that lies in the space $(\mathfrak{F}, P)$. This class of sets is defined as: $\mathfrak{F}^{(u)} = \{A' \subset \Omega' : u^{-1}[A'] \in \mathfrak{F}\}$. We can assign probabilities to the sets in $\mathfrak{F}^{(u)}$ by: $\forall A' \in \mathfrak{F}^{(u)}, \quad P^{(u)}(A') = P(u^{-1}[A'])$. This function $P^{(u)}(A')$ is called the **probability function of** u. Note that this idea of the pre-image of u being in $\mathfrak{F}$ is analogous to the concept of measurable function in analysis, then a probability function is a measurable function. Moreover, given a probability function u, we can find a partition: $\mathfrak{U} = \{u^{-1}(i) : i \in I\}$, where $u^{-1}(a) = \{\omega \in \Omega : u(\omega) = a\}$. The next theorem, stated in [39] and proved in [1], shows (among other results) that $\mathfrak{F}^{(u)}$ is a σ-algebra.

Theorem 5.3.1. *$(\Omega', \mathfrak{F}^{(u)}, P^{(u)})$ is a probability space.*

Proof. To see that $(\Omega', \mathfrak{F}^{(u)}, P^{(u)})$ is a probability space, we need to show that all of the six axioms hold for this space.

Axiom I: $\mathfrak{F}^{(u)}$ is a σ-algebra over Ω'.

$\mathfrak{F}$ is a σ-algebra over Ω, then the pre-image commutes with the operations of complement and countable unions, so $\mathfrak{F}^{(u)}$ is also closed under these operations and $\mathfrak{F}^{(u)}$ is a σ-algebra over Ω'.

Axiom III: To each set $A' \in \mathfrak{F}^{(u)}$, note that $P(A')^{(u)}$ is a non-negative number by construction.

Axioms II and IV: $\mathfrak{F}^{(u)}$ contains Ω', and $P(\Omega')^{(u)} = 1$ because $u^{-1}[\Omega'] = \Omega$.

Axioms V and VI: we will show that countable additivity holds, so we get the finite additivity and, by the theorem 5.2.2, we also get the countable additivity.

Let's take a countable collection of pairwise disjoint subsets of Ω: $A_1, A_2, \ldots$. We have that $u^{-1}(\cup_i A_i) = \cup_i u^{-1}[A_i]$ where the $u^{-1}[A_i]$'s are pairwise disjoint.

$$\begin{aligned} P^{(u)}(\cup_i A_i) &= P(u^{-1}[\cup_i A_i]) \\ &= P(\cup_i u^{-1}[A_i]) \\ &= \sum_i P(u^{-1}[A_i]), \text{ because the pre-images commute with disjoint unions} \\ &= \sum_i P^{(u)}(A_i) \text{ by countable additivity of } P \text{ and } u^{-1}(\cup_i A_i) = \cup_i u^{-1}[A_i]. \end{aligned}$$

□

This concept of a field of probability is essential in eliminating ambiguities from the classical approach and as a consequence, overcoming many epistemological obstacles. It offers a formal

construction to model random experiments and any well-posed question about the probability of an event A must be in unique correspondence to a question about the probability of a set A. Two formal calculations can't result in different answers, because the probability space, by definition, specifies uniquely the probabilities of all events.

Given a probability space over the domain of u, we have induced the probability space over its image, so $P^{(u)}(A') = P(u(a) \in A') = P(u^{-1}[A'])$. Now we formalize the concept of a random variable according to Kolmogorov. Using the results of measurability from Lebesgue, it is now defined as a measurable function. That is: a real function $x(\xi)$ defined on Ω is called a **random variable** if, for each choice of a real number a, the set $\{x < a\}$ of all ξ for which the inequality $x < a$ holds true, belongs to the σ-algebra $\mathfrak{F}$.

5.3.2 Mathematical expectation

In this subsection we will apply Lebesgue's integral to random variables in order to define mathematical expectation. Let's consider a probability space $(\Omega, \mathfrak{F}, P)$, a random variable $x : \Omega \to \mathbb{R}$, and $A \in \mathfrak{F}$. Let's also take $\mathfrak{U}$ as a partition of A into sets B, x_B the values that $x(\omega)$ takes for $\omega \in B$. Our goal is to approximate the integral of x over A by sums $\sum_{B\in\mathfrak{U}} x_B P(B)$, because a random variable is now defined as a measurable function and its expectation is defined as the Lebesgue integral.

Even though $A \in \mathfrak{F}$, the partition $\mathfrak{U}$ of A is not defined in the domain of x. Instead, we take a partition of image of x, that is the real line, into intervals $[k\lambda, (k+1)\lambda)$ and construct the partition $\mathfrak{U}$ considering the inverse images of these intervals. As x is a measurable function by definition, by taking its pre-image to construct the partition, we guarantee the measurability of $\mathfrak{U}$. This is the principle used to construct the Lebesgue integral [1].

We take the series: $S_\lambda(x, A, P) = \sum_{k=-\infty}^{k=+\infty} k\lambda P(\{\omega : k\lambda \leq x(\omega) < (k+1)\lambda\} \cap A)$. If it converges absolutely for every λ, and its limit exists when $\lambda \to 0$, then it is the Lebesgue integral of x over A, relative to the probability measure P: $\lim_{\lambda\to 0} S_\lambda = \int_A x(\omega) d_{P(\omega)}$. If we take the integral of x over the whole space Ω, we have the **mathematical expecation** of the random variable x:

$$E(x) = \int_\Omega x(\omega) d_{P(\omega)}.$$

5.3.3 Conditional probability

In his chapter on elementary probability, Kolmogorov defined the conditional probability of event B under the condition of event A as the unique solution of: $P_A(B) = \frac{P(A \cap B)}{P(A)}$ whenever $P(A) > 0$.

It's remarkable that this definition is valid only when $P(A) > 0$, however, $P(A) = 0$ doesn't mean that the event A is impossible. Some paradoxes, like the Buffon's Needle problem[6] or the the great circle problem[7], led to paradoxical results when the solution is based on the classical approach to conditional probability. It was necessary to generalize this concept in order to be able to handle many common situations where we need to impose a condition on probability 0 events.

As an example, let's consider a two step experiment, where a random variable Y is observed after the random variable X, so the distribution of Y depends on the value x of X. Let $x : 0 \leq x \leq 1$ be the probability landing on heads in a coin toss and Y be the number of heads in n independent coin tosses. Then, $P\{Y \in B | X = x\}$, the probability of Y given $\{X = x\}$, may have probability 0 for all values of x. Intuitively, we know that $P(Y = k | X = x) = \binom{n}{k} x^k (1 - x)^{n-k}$. With Kolmogorov's approach we are able to define probability conditional on a choice out of a partition of Ω indexed by an *arbitrary* set I or on the value of a probability function. The development of probability conditional to measure zero sets was only made possible, as we will show in this section, by the generalization achieved in the Radon-Nikodyn theorem. In this section on conditional probability, we will expose the results as in Kolmogorov's book and in the next section, *Expectation conditional to a σ-algebra*, we will present an analogue result of conditional expectation as it is presented in modern literature. The exposition that follows is based on [39] as well as in some demonstrations developed in [1].

Any random variable $P_u(B)$ that satisfies (4) is called **a version of the conditional probability of B with respect to the partitioning u**:

$$\forall C \in \mathfrak{F}^{(u)}, \quad P(u^{-1}(C) \cap B) = \int_C P_u(B)(a) dP^{(u)}(a) \tag{4}$$

Note that $P_u(B)$ must be a random variable so we can have the Lebesgue integral defined. We want to prove the existence and uniqueness of that random variable, but as we are talking about random variables, or integrable functions, we can only state the uniqueness up to equivalence

[6] From chapter four.
[7] It will exposed in the next pages.

classes. That is why it is called a **version** of the conditional probability. This means that any two versions will be equal for all $a \in u[A]$, except on a set $C \in \mathfrak{F}^{(u)}$ with $P^{(u)}(C) = 0$. We will recall the Radon-Nikodym theorem in order to prove the following two theorems, that will show the existence of the random variable, and a third which will prove the uniqueness up to equivalence classes.

Theorem 5.3.2 (Radon-Nikodym)**.** *Let μ and λ be σ-finite measures on a σ-algebra $\mathcal{A}$ associated to a set S such that: $\forall C \in \mathcal{A}, \quad \lambda(C) \neq 0 \Rightarrow \mu(C) \neq 0$, that is, $\lambda \ll \mu$. Then, $\lambda = f\mu$ for some non-negative Borel function $f : S \to \mathbb{R}$, and $\lambda = f\mu$ means $\lambda(C) = \int_C f(a)d\mu(a)$.*

Theorem 5.3.3. *There always exists a random variable $P_u(B)$ that satisfies (4).*

Proof. We use the Radon-Nikodym theorem to prove this theorem, so we need to show that the conditions of the Radon-Nikodym theorem hold for our definition of $P_u(B)$. Note that: $S = u[\Omega]$, $\mathcal{A} = \mathfrak{F}^{(u)}$, $\mu = P^{(u)}$ and

$$\lambda : C \to P(B \cap u^{-1}[C]), \quad (C \in \mathfrak{F}^{(u)}). \tag{5}$$

Probability measures are, by definition, finite, hence σ-finite. λ is a measure because inverse images commute with all set operations, so λ inherits countable additivity from P. Finally, equation (5) holds because: $\forall C \in \mathfrak{F}^{(u)} : P(B \cap u^{-1}[C]) = \lambda(C) = \int_C f(a)d\mu(a) = \int_C f(a)dP^{(u)}(a)$ for some non-negative random variable f, which gives us the existence. □

Theorem 5.3.4. *Any two random variables like $P_u(B)$ are equal almost everywhere.*

Proof. Let's consider any two random variables $x : u[\Omega] \to \mathbb{R}$ and $y : u[\Omega] \to \mathbb{R}$, both satisfying (4) for any $C \in \mathfrak{F}^{(u)}$. Then we get the equivalence by:

$$\int_C x(a)dP^{(u)}(a) = \int_C y(a)dP^{(u)}(a) = P\left(B \int u^{-1}(C)\right).$$

□

Now we just need two theorems to show that $P_u(B)(a)$ as a function of (B) satisfies the axioms of probability almost everywhere.

Theorem 5.3.5. $0 \leq P_u(B) \leq 1$ *almost everywhere.*

Proof. Using Radon-Nikodym's theorem, we see that $0 \leq P_u(B)$ by noting that $P_u(B)$ is almost everywhere equal to f, which is non-negative.

Suppose that there exists some $M \in \mathfrak{F}$ such that $P^{(u)}(M) > 0$ and $P_u(B)(a) > 1$ for every a in M. Now we have:

$$P_u(B)(a) > 1 \Leftrightarrow \exists n, P_u(B)(a) \geq 1 + 1/n$$

$$M \subset \cup_n M_n, \text{ where } M_n = \{a : P_u(B)(a) \geq 1 + 1/n\}$$

Hence, $P^{(u)}(M_k) > 0$ for at least one natural number k, otherwise, $P^{(u)}(M) \leq P^{(u)}(\cup_{n\in\mathbb{N}} M_n) = \sum_n P^{(u)}(M_n) = 0$, which contradicts the hypothesis that $P^{(u)}(M) > 0$. Now let's set $M' = M_k$.

$$\begin{aligned}
P(B \cap u^{-1}(M')) &\geq P_{u^{-1}(M')}(B) \text{ by elementary conditional probability} \\
&= E_{u^{-1}(M')}(P_u(B)) \text{ by (4)} \\
&\geq E_{u^{-1}(M')}(1 + 1/n) \\
&= (1 + 1/n)P(u^{-1}(M')) \text{ by the definition of expectation} \\
&> P(u^{-1}(M')), \text{ which is a contradiction.}
\end{aligned}$$

□

Theorem 5.3.6. *If $B = \cup_{n\in\mathbb{N}} B_n$, a union of pairwise disjoint sets, then $P_u(B) = \sum_n P_u(B_n)$.*

Proof. Note that if $C = u[\Omega]$ in (4), we get:

$$P(B) = E(P_u(B)) \tag{6}$$

Now we can write:

$$\begin{aligned}
P(B) &= \sum_n P(B_n) \text{ by countable additivity of } P \\
&= \sum_n E(P_u(B_n)), \text{ by (6)} \\
&= \sum_n E(|P_u(B_n)|), \text{ since } P_u(B_n) = |P_u(B_n)| \text{ almost everywhere.}
\end{aligned}$$

$\sum_n E(|P_u(B_n)|)$ converges because $P(B)$ is finite. So, for any $C \in \mathfrak{F}^{(u)}$ such that $P^{(u)}(C) >$

0, we get:

$$\begin{aligned} E_{u^{-1}[C]}(P_u(B)) &= P_{u^{-1}}(B) \text{ by (4)} \\ &= \sum_n E_{u^{-1}[C]}(P_u(B_n)), \text{ by additivity of } P \text{ and (4)} \\ &= E_{u^{-1}[C]}(\sum_n P_u(B_n)) \end{aligned}$$

because $\sum_n E(|P_u(B_n)|)$ converges. But this implies that $\sum_n P_u(B_n) = P_u(B)$ almost everywhere by the same proof as the uniqueness almost everywhere in theorem (5.3.4). □

To finish this subsection, we will provide an example to illustrate probability conditional to measure zero sets. It is a simple case where the classical approach can't handle because it requires the conditional event to have strictly positive probability and Kolmogorov's approach handles it without ambiguities.

Let $X \sim U[0,1]$ represent the probability of heads in a coin toss. Let Y be the number of heads after n independent coin tosses. Find $P\{Y = k\}, k = 0, 1, \ldots, n$.

Solution: Let $\Omega_1 = [0,1]$, $\mathfrak{F}_1 = \mathfrak{B}[0,1]$, $\Omega_2 = \{0, 1, \ldots, n\}$, $\mathfrak{F}_2$ be the set of all subsets of Ω_2. $P_X(A) = \int_A dx$ is the Lebesgue measure of A, $A \in \mathfrak{F}_1$.

For each x, $P(x, B)$ is the conditional probability that $Y \in B$, given $X = x$. $P(x, \{k\}) = \binom{n}{k} x^k (1-x)^{n-k}$, $k = 0, 1, \ldots, n$, is measurable in x. Now we set $\Omega = \Omega_1 \times \Omega_2$, $\mathfrak{F} = \mathfrak{F}_1 \times \mathfrak{F}_2$ and P is the probability measure:

$$P(C) = \int_0^1 P(x, C(x)) dP_x(x) = \int_0^1 P(x, C(x)) dx.$$

Now, let $X(x, y) = x$ and $Y(x, y) = y$.

$$\begin{aligned} P\{Y = k\} = P(\Omega_1 \times \{k\}) &= \int_0^1 P(x, \{k\}) dx \\ &= \int_0^1 \binom{n}{k} x^k (1-x)^{n-k} dx = \binom{n}{k} \beta(k+1, n-k+1), \end{aligned}$$

where $\beta(r, s) = \int_0^1 x^{r-1}(1-x)^{s-1} dx$, $r, s > 0$, is the beta function. Expressing $\beta(r, s) =$

$\Gamma(r)\Gamma(s)/\Gamma(r+s)$, with $\Gamma(n+1) = n!$, we can conclude that:

$$P\{Y = k\} = \frac{\binom{n}{k}k!(n-k)!)}{(n+1)!} = \frac{1}{n+1}, \quad k = 0, 1, \ldots, n.$$

5.3.4 Expectation conditional to a σ-algebra

We've just described *conditional probability* when we consider a partition of Ω with an arbitrary index set I. In modern literature, this concept is introduced as a random variable called **probability conditional to a σ-algebra**. This random variable is obtained by the expectation of a characteristic function over a set, conditional with respect to a σ-algebra. There is no substantial theoretical innovation vis-à-vis the previous section, once the main changes here are the modern notation and the conditioning to a σ-algebra, instead of an arbitrary partition. The theorems are also afforded by the Radon-Nikodym theorem from chapter four. The results presented in this subsection are based on [59] and [2].

Let Y be a random variable with finite expectation defined on $(\Omega, \mathfrak{F}, P)$. Now we take the functions: $X : (\Omega, \mathfrak{F}) \to (\Omega', \mathfrak{F}')$, $g : (\Omega', \mathfrak{F}') \to (\mathbb{R}, \mathfrak{B})$ and $h : (\Omega, \mathfrak{F}) \to (\mathbb{R}, \mathfrak{B})$, such that $h(\omega) = g(X(\omega))$. Thus $h(\omega)$ is the conditional expectation of Y, given that X takes the value $x = X(\omega)$. Consequently, h measures the average of Y given X, but h is defined on Ω, instead of Ω'.

Note that:

$$\int_{X\in A} h dP = \int_{\Omega} g(X(\omega))\mathbb{I}_A(X(\omega))dP(\omega) = \int_{\Omega'} g(x)\mathbb{I}_A dP_X(x) = \int_A g(x)dP_X(x) = \int_{X\in A} Y dP$$

.

Since $\{X \in A\} = X^{-1}(A) = \{\omega \in \Omega : X(\omega) \in A\}$, we may set $X^{-1}(\mathfrak{F}') = \{X^{-1}(A) : A \in \mathfrak{F}'\}$, the σ-algebra induced by X. So we can state that, for each $C \in X^{-1}(\mathfrak{F}')$, we have $\int_C h dP = \int_C Y dP$. Now we can define the conditional expectation given a σ-algebra.

Let Y be an integrable random variable on $(\Omega, \mathfrak{F}, P)$, $\mathcal{G}$ a sub σ-algebra of $\mathfrak{F}$. A function $E(Y|\mathcal{G}) : (\Omega, \mathcal{G}) \to (\mathbb{R}, \mathfrak{B})$ that is $\mathcal{G}$-measurable and:

$$\int_C Y dP = \int_C E(Y|\mathcal{G})dP, \text{ for each } C \in \mathcal{G},$$

is called **the conditional expectation of Y given $\mathcal{G}$**.

The existence and uniqueness up to equivalence classes of the function $E(Y|\mathcal{G})$ can be proven exactly in the same way that we've done in theorems (5.3.3) and (5.3.4).

If we set $X : (\Omega, \mathfrak{F}) \to (\Omega, \mathcal{G})$ to be the identity map: $X(\omega) = \omega$, $\omega \in \Omega$, then we have $X^{-1}(\mathcal{G}) = \mathcal{G}$, $g(x) = E(Y|X = x)$ and $h = E(Y|\sigma - (X)) = E(Y|\mathcal{G})$. In order to bring some intuition into this discussion, we can think of $E(Y|\mathcal{G})$ as $E(Y|X)$, that is, the average value of Y given that $X : (\Omega, \mathfrak{F}) \to (\Omega, \mathcal{G})$ is known. The random variable X is composed of sets of the form $\{X \in G\}$, $G \in \mathcal{G}$, because $\{X \in G\} = G$ (X is the identity map). So $E(Y|\mathcal{G})$ can be thought of as the average of $Y(\omega)$, provided we know whether or not $\omega \in G$, for each $G \in \mathcal{G}$.

As an example, let's take the random variables X and Y with joint density f. Let $\Omega = \mathbb{R}^2$, $\mathcal{G} = \mathfrak{B}(\mathbb{R}^2)$, $P(B) = \int\int_B f(x,y)dxdy$, $B \in \mathfrak{F}$, $X(x,y) = x$ and $Y(x,y) = y$. Also let's set $\Omega' = \mathbb{R}$, $\mathfrak{F}' = \mathfrak{B}(\mathbb{R})$.

$g(x) = E(Y|X = x) = \int_{-\infty}^{\infty} yh_0(y|x)dy$, where h_0 is the conditional density of Y given X. Let $h = E(Y|X)$, that is, $h(\omega) = g(X(\omega)$.

We can see that $E(Y|X)$ is constant on vertical strips, $X^{-1}(\mathfrak{F}')$ consists of all sets $B \times \mathbb{R}$, $B \in \mathfrak{B}(\mathbb{R})$. Since $x \in B \Leftrightarrow (x,y) \in B \times \mathbb{R}$, the information about $X(\omega)$ is equivalent to the information whether or not $\omega \in \mathcal{G}$.

To close this section, we will show that the probability of a set conditional to a σ-algebra can be obtained by the expectation conditional to that σ-algebra.

Theorem 5.3.7. *Let $(\Omega, \mathfrak{F}, P)$ be a probability space, $\mathcal{G} \subset \mathfrak{F}$ and fix $B \in \mathfrak{F}$. There is a $\mathcal{G}$-measurable function $P(B|\mathcal{G}) : (\Omega, \mathcal{G}) \to (\mathbb{R}, \mathfrak{B})$, called **the conditional probability of B given $\mathcal{G}$**, such that*

$$P(C \cap B) = \int_C P(B|\mathcal{G})dP, \text{ for each } C \in \mathcal{G}.$$

Proof. The existence and uniqueness up to equivalence classes is shown exactly as in theorems (5.3.3) and (5.3.4). □

The probability conditional to a σ-algebra is the expectation of a random variable conditional to a σ-algebra but instead of using the random variable Y as we did for the conditional expectation, we use characteristic function over the set B, $\mathbb{I}_B$.

So far, we have discussed conditional probability defined up to equivalence classes. However, what happens if the set of points where the conditional probability is not defined is uncountable? That is, what happens if the set of ω's where countable additivity fails is uncountable? This last

question is not treated in Kolmogorov's book, but is presented in most modern texts on probability. References of the authors who discussed this term after Kolmogorov's book is found in [42].

If $B_1, B_2, \ldots$ are pairwise disjoint sets in $\mathfrak{F}$, then $P(\cup_{n=1}^{\infty} B_n|\mathcal{G}) = \sum_{n=1}^{\infty} P(B_n|\mathcal{G})$ almost everywhere[8]. This equation is only satisfied almost surely, that is, up to equivalence classes. Consequently the conditional probability $P(B|\mathcal{G})(\omega)$ cannot be considered a measure on B for given ω. Now let's take the set $N(B_1, B_2, \ldots)$ where countable additivity fails for a given ω. Now, for all given ω's the set where countable additivity fails is given by $M = \cup N(B_1, B_2, \ldots)$. As M is an uncountable union of sets, it may not have probability 0, even though each set N has probability 0. The following definition solves this inconvenience by setting conditions by which the conditional probability $P(\cdot|\mathcal{G})(\omega)$ is a measure for each ω.

A function $P(\omega; B)$, defined for all $\omega \in \Omega$ and $B \in \mathfrak{F}$, is a **regular conditional probability** with respect to $\mathcal{G} \subset \mathfrak{F}$ if:

i) $P(\omega; \cdot)$ is a probability measure on $\mathfrak{F}$ for every $\omega \in \Omega$, and

ii) For each $B \in \mathfrak{F}$, the function $P(\omega; B)$, as a function of ω, is a version of the conditional probability $P(B|\mathcal{G})(\omega)$, that is: $P(\omega; B) = P(B|\mathcal{G})(\omega)$ almost surely.

5.4 The great circle paradox

In this section, we want to introduce a paradox from classical probability, called the great circle paradox, or Borel's paradox, as an example of how Kolmogorov's work established the ground for a formal development of probability theory free of ambiguities. This paradox was published by Bertrand [8] and is addressed in Kolmogorov's book [39]. Bertrand stated the problem as: "*On fixe au hasard deux points sur la surface d'une sphère; quelle est la probabilité pour que leur distance soit inférieure à 10'?*[9]" [8] (p. 6).

In this problem, two points are randomly chosen with respect to the uniform distribution on the surface of a unit sphere and we want to find the probability that the distance between them will be less than 10'. We can find two solutions using the classical approach in this problem. The first one is to calculate the proportion of the sphere's surface area that lies within 10' of a given point, let's say, the north pole, see part a of Figure (5.3). Another solution is that there exists a unique great circle that connects the second random point to the north pole. Each great circle is

[8] We will not prove this result here, but the interested reader can consult [2]

[9] By 10' we mean 1/6 of a degree. 1 degree = 60'.

equally likely to be chosen, so the problem has been reduced to finding the proportion of the length of the great circle that lies within 10' from the north pole, see part b of Figure (5.3). These two solutions are intuitively equivalent, but the tension arises because these solutions lead to different results. Since a great circle has measure zero on a sphere, the classical formula for the conditional probability from Bayes cannot be used to calculate the conditional probability in question.

In the first solution, the area of the sphere's cap that lies within 10' from the North Pole is given by: $2\pi r^2(1-\cos\theta) = 2\pi(1-\cos(1/6))$. The area of the whole sphere is given by $4\pi r^2 = 4\pi$. The probability is given by: $\frac{2\pi(1-\cos(1/6))}{4\pi} \approx 2.1 \times 10^{-6}$.

In the second solution, the arc length on the sphere is given by the formula: $l = \frac{r\pi\theta}{180}$. The arc length for a distance given by 10' is: $l = \frac{\pi}{6\times 180} = \frac{\pi}{1080}$ and the length of one great circle is 2π. It's important to remember that we need to consider twice the arc length, because on one great circle, starting from the North Pole, we can have two arcs given by 10'. The probability is given by: $\frac{2\pi}{2\pi 1080} \approx 9.3 \times 10^{-4}$.

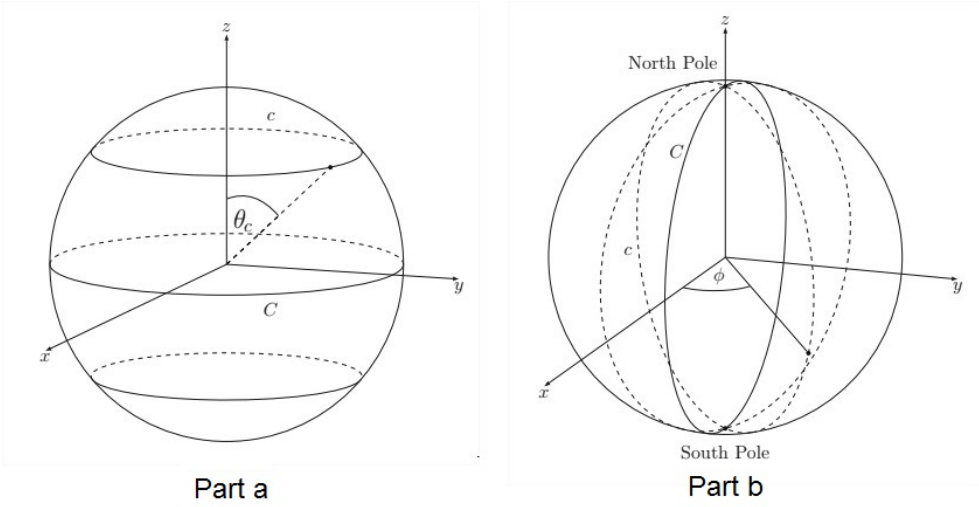


Figure 5.3: The great circle paradox - [29] (p. 2612 and 2614).

To solve the great circle paradox, we will write all relevant information according to Kolmogorov's formalization as described in [1]. Look at Figure (5.4) and set:

- $\Omega = \{$the set of points of a unit sphere $\} \subset \mathbb{R}^3$;
- $\mathfrak{F} = \{$Borelian sets on $\Omega\}$;
- $P(A)$ the lebesgue measure of A, $Leb(A)$, $(A \in \mathfrak{F})$;

- φ, the polar angle of the vector ξ from the positive z-axis in the range $[0, \pi]$ - the co-latitude of ξ;
- λ, the angle between the projection of the vector ξ in the equatorial plane and the positive x-axis, with range $[0, 2\pi)$, measured clockwise.

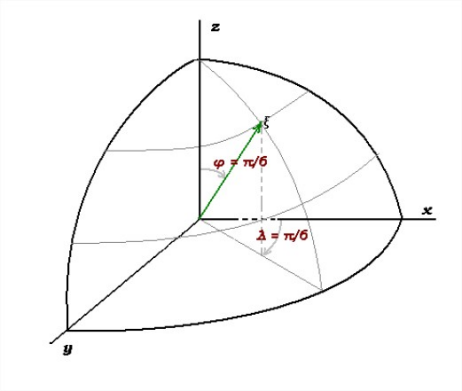


Figure 5.4: Parametrization of the sphere - [1] (p. 83).

φ and λ are probability functions carried by $(\Omega, \mathfrak{F}, P)$, because the pre-images under φ or λ are intersections of Borelian sets in $\mathbb{R}^3$. We can construct the bi-dimensional probability space $(\mathbb{R}^2, \mathfrak{B}(\mathbb{R}^2), P^{(u)})$, where $u(\xi) = (\varphi(\xi), \lambda(\xi))$.

Let $\varphi_1, \varphi_2 \in [0, \pi]$ and $\lambda_1, \lambda_2 \in [0, 2\pi]$. Then,

$$\begin{aligned} P^{(u)}([\varphi_1, \varphi_2] \times [\lambda_1, \lambda_2]) &= Leb(\{\xi | \varphi_1 \leq \varphi(\xi) \leq \varphi_2, \lambda_1 \leq \lambda(\xi) \leq \lambda_2\}) \\ &= \int_{\varphi_1}^{\varphi_2} \int_{\lambda_1}^{\lambda_2} \sin(\varphi) d\lambda d\varphi = \frac{1}{4\pi}(\lambda_2 - \lambda_1)(\cos(\varphi_1) - \cos(\varphi_2)). \end{aligned}$$

Without loss of generality, let's fix the first point as the North Pole. Then we can call B the event that we are interested in: $B = \{\xi | \varphi(\xi) < c\}$. In this case, $c = 10' = 2\pi/(6 \cdot 360) = \pi/1080$.

The set B is given by the pre-image $u^{-1}([\varphi_1, \varphi_2] \times [\lambda_1, \lambda_2]) = u^{-1}([0, c] \times [0, 2\pi])$. So we get:

$$P(B) = P^{(u)}([0, c] \times [0, 2\pi]) = \frac{1}{4\pi}(2\pi - 0)(\cos(0) - \cos(c)) = \frac{1 - \cos(c)}{2} = 2.115397 \times 10^{-6}.$$

5.4.1 Closing remarks from the great circle paradox

Now we must ask ourselves: what is wrong with the solution that gave us 9.3×10^{-4}? This result is based on symmetry, that is, the probability of B is not affected by conditioning on a choice of half-meridian, which is true. Nevertheless, it also considered this probability as a 1-dimensional Lebesgue measure, that is, the the intersection of B with the half-meridian:

$$P_\lambda(\lambda_0; B) = Leb(B \cap \{\xi : \lambda(\xi) = \lambda_0\}) = c/2\pi. \tag{7}$$

The problem arises because the symmetry and the equation (7) are not compatible, since the value of $P(B)$ is given by the choice of initial probability space and cannot be revised midway and transformed from a 2 to a 1-dimensional space. So the main problem with this reasoning is the revision of the probability space midway.

By using symmetry, we have that $P(B)$ must be equal to $P_\lambda(\lambda_0; B)$. However, the probability of B conditional on a choice of half-meridian is not the Lebesgue measure of the arc. If we set independent grounds in favour of (7) we can write:

$$P(\xi \in B|\lambda(\xi) = \lambda_0) = \frac{P(B \cap \{\xi|\lambda(\xi) = \lambda_0\})}{P(\{\xi|\lambda(\xi) = \lambda_0\})} = \frac{P(\text{arc})}{P(\text{half-meridian})}.$$

A naïve approach leads P to be the 1-dimensional Lebesgue measure, $\frac{Leb(\text{arc})}{Leb(\text{half-meridian})} = \frac{c}{2\pi}$, however P is actually the 2-dimensional Lebesgue measure. This approach fails because it doesn't formalize the space as a 2-dimensional one and uses a 1-dimensional Lebesgue integral to evaluate the probability after using symmetry.

The paradoxes from Bertrand and geometric probability can be effectively solved with Kolmogorov's approach. In fact, the probability space as conceived by Kolmogorov offers a formal construction to model random experiments and "*Any well-posed question about the probability of an event A must be in unique correspondence to a question about the probability of a set A that is represented in a specified probability space*" (p. 7). Probabilities, in the modern approach are assignments on *sets* of elementary outcomes, and in the classical one they are assignments on elementary outcomes themselves.

In finite probability spaces, these two approaches are equivalent because singletons are uniquely extended to assignments on arbitrary, but in infinite sample spaces, that is not the case and it puts

in evidence the limitations of the equiprobability assumption (or the *principle of indifference*).

When two formal calculations result in different answers, the probability space should be carefully looked into, because, by definition, the probability space specifies uniquely the probabilities of all events.

Chapter 6

Book Analysis

6.1 Introduction

In this chapter we analyze five probability textbooks that are commonly adopted by universities in Montreal from 1^{st} year undergraduate up to graduate level in mathematics and statistics courses. As discussed in chapter 1, our interest lies in investigating which approach, classical or modern, is primarily advance by undergraduate and graduate texts in probability. For this purpose, we consider how they introduce the concept of probability and the exercises and examples they proposed around the definition of this concept. Do the examples stimulate some reflection on the modern and axiomatic definition of probability or do they just touch on slightly, focusing primarily on other mathematical concepts, such as counting techniques, set operations or measure theory? Do the sets of exercises make students think about Kolmogorov's innovation or do they stimulate the idea of probability as a proportion of favourable over possible cases? In other words, we are interested in analyzing how the books introduce the concept of probability and whether they foster a framework that promotes the modern approach to probability, or whether they continue to favor the classical approach, or other mathematical concepts.

In the next section we describe the method for analyzing the books. The second section analyzes each book with a brief expose of its intended level (graduate or undergraduate) and discusses: i) the context, the definition and the examples around probability, ii) the set of exercises and iii) the way each book enhances or weakens a modern thinking of probability. We conclude the chapter comparing the five books.

6.2 Methodology

To do the analysis of the books as proposed in the introduction of this chapter we adopted a three step procedure: i) a book selection, ii) a characterization of the book, which includes a description of the book level as well as the context around the definition of probability and iii) an analysis of the set of exercises.

6.2.1 The book selection

The books analyzed were chosen from available probability course outlines of the graduate and undergraduate level from each of the four universities in Montreal. Also, even though it is not mentioned in any of the course outlines, we decided to analyze the book written by Shiryaev (2016). We've chosen this book because Shiryaev was very familiar with Kolmogorov's work and view of probability, given that he was Kolmogorov's direct student and has organized, edited and commented on his work. The books we selected to analyze were:

(1) Wackerly, D. D. ; Mendenhall, W. and Scheaffer, R. L. Mathematical Statistics with Applications, 7^{th} Edition, Duxbury Press, 2007;

(2) Ross, S. A first course in probability, 8^{th} edition. Prentice Hall, 2010;

(3) Grimmet, G. R. and Stirzaker, D. R. Probability and Random Processes, Oxford 2001;

(4) Shiryaev, Probability 1, 3^{rd} edition. Springer, 2016;

(5) Durrett, R. Probability: Theory and Examples, 4^{th} edition, Cambridge University Press, 2010.

The first two books are used in undergraduate level courses, the third and fourth books are used either in an advanced undergraduate level course or in a graduate course and the fifth book is used in graduate courses. We are aware that the books may have different editions, but the changes are rather minor and there isn't any substantial modification in the approach, content exposition, or set of exercises.

6.2.2 The characterization of the books

The books are characterized according to two categories, the first being the level of the book and the second, the context in which the definition of probability is presented. To be more specific,

the first category concerns the public addressed by the book - undergraduate or graduate students, the pre-requirements established by the authors in terms of mathematical disciplines or knowledge and whether the book proposes a theoretical or an applied approach. The second category concerns the context in which probability is defined. We are interested in the discussions and examples that introduce and/or explain the definition of probability. Our characterization is guided by questions such as: Do these discussions and examples stimulate thinking in terms of a modern and axiomatic approach? Do the examples illustrate any difference between modern and classical probability? Is there any discussion to show that the examples satisfy the axiomatic definition of probability? What are the aspects of modern probability that are illustrated by the examples?

6.2.3 Analysis of the set of exercises

Exercises and problems play a fundamental role in learning mathematics, because they illustrate and help developing problem solving strategies and an engagement in mathematical practices that are fundamental aspects of thinking mathematically [55]. We have read all the exercises of the sections of the book where the concept of probability is introduced and we analyze them with the goal of identifying what type of knowledge or skills they require to be solved. We want to see whether the proposed tasks are connected to the definition of probability or not. Which approach to probability do the exercises enhance: a classical or a modern one? Do the exercises call attention to situations where the classical approach doesn't work? Do they promote any thinking on the innovations brought by Kolmogorov? More specifically, we want to find whether or not the books treat infinite probability spaces, countable additivity, situations where not every single event can have a probability measure and the necessity of an axiomatic approach.

After reading the exercises, we classified each of them into one or more of the categories below and regrouped them into Venn diagrams to obtain a visual presentation of the type of knowledge or skills they require to be solved. We are aware that more than one solution might be possible for a task and that the classification provided below sorted exercises into categories that we ourselves have chosen, and relies on our own interpretation of those exercises. For this reason, this classification cannot be said to be unique. The categories of knowledge or skills necessary to solve the problems are the following:

- **Finite ss:** finite sample space.

- **Infinite ss:** infinite sample space.
- **Irrelevant ss:** the sample space's size, whether finite or infinite, is not relevant in the exercise resolution.
- **Prop reasoning:** probability is associated with the proportion of favorable over possible cases.
- **Counting:** a counting strategy, either from combinatorial analysis or any informal one.
- **Set operations:** set operations, such as taking unions or intersections of sets.
- **Additivity:** finite or countable additivity and/or that the probability of the whole sample space equals to 1.
- **Cond prob or indep:** the concepts of conditional probability or independence.
- **Convergence:** modify an expression and explore the convergence results.
- **Measure:** results or definitions exclusively from measure theory associated with little (or no) knowledge of probability.
- **Def of prob:** the axiomatic definition of probability.
- **Sigma-field:** the definition of a sigma-field.

The labels are used in the Venn diagrams where we classify the exercises of the textbooks.

6.3 Book analysis

After presenting the method for selecting and characterizing the books, analyzing the context of the definition of probability and the set of exercises, we present in this section the discussion on each book.

6.3.1 Book 1: Wackerly, D. D. ; Mendenhall, W. and Scheaffer, R. L. Mathematical Statistics with Applications [68].

In the preface of the book, the authors indicate that the intent of the book is to provide a solid undergraduate foundation in statistical theory and highlight the importance of theory in solving practical problems. The mathematical pre-requisite is a 1^{st} year calculus course.

The context

The authors introduce the concept of probability in an informal and intuitive way in the first paragraph of the second chapter as: "*In everyday conversation, the term probability is a measure of one's belief in the occurrence of a future event*" (p. 20). In what follows, the authors present the relative frequency as a way to evaluate probabilities: "*This stable long-term relative frequency provides an intuitively meaningful measure of our belief in the occurrence of a random event if a future observation is to be made*" (p. 20).

They mention that relative frequency doesn't provide a rigorous definition of probability, but they don't develop this any further. It is not mentioned what a rigorous definition of probability would be, why relative frequency doesn't provide such a definition and what the problems are with that approach. They only say: "*Nevertheless, for our purposes we accept an interpretation based on relative frequency as a meaningful measure of our belief in the occurrence of an event*" (p. 21).

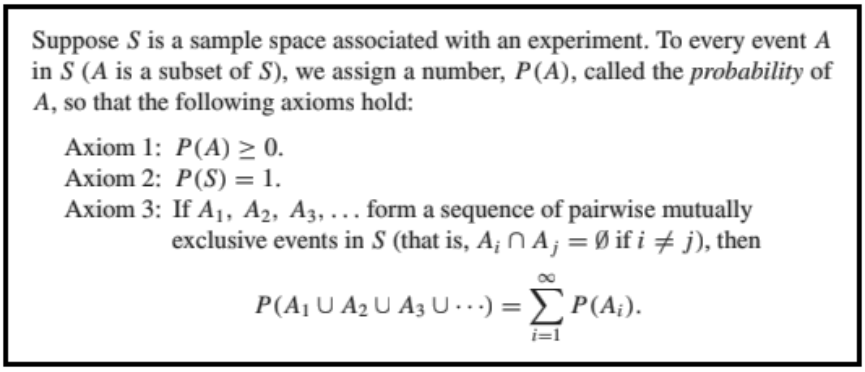
The second section presents an example to develop the intuition of the importance of probability in statistical inference, but our main interest lies in the fourth section. There, the authors present the definition of experiment, event and sample space with an example of a finite (a die throw) and an approximation of an infinite sample space (a bacteria population).

In what follows, an explanation for the option to use relative frequency is provided: "*Although relative frequency does not provide a rigorous definition of probability, any definition applicable to the real world should agree with our intuitive notion of the relative frequencies of events* (p. 29)." The problem with this justification is that it constrains real world situations where probability can be applied, to those who fit the relative frequency approach. This denies many real world problems that had partially motivated the modern and axiomatic approach to probability, such as the problems coming from quantum mechanics, continuous time Markov chains, Brownian motion and even the famous 6^{th} Hilbert problem as we have shown in the fourth chapter of this thesis. Instead of Kolmogorov, the founder of modern probability would be von Mises. This statement misleads students into an oversimplification of the concept of probability and the problems it can resolve.

Before presenting the axiomatic definition (in section 2.4), the authors present an intuitive and informal discussion to show that the relative frequency fits into the axioms of probability by saying that: i) the relative frequency of the whole space is 1; ii) it is non-negative and; iii) the relative frequency of the union of two mutually exclusive events is the sum of their respective

relative frequencies.

By these three items, the authors are preparing the reader to be introduced to the axiomatic definition of probability as in the Figure (6.1).



Suppose S is a sample space associated with an experiment. To every event A in S (A is a subset of S), we assign a number, $P(A)$, called the *probability* of A, so that the following axioms hold:

Axiom 1: $P(A) \geq 0$.
Axiom 2: $P(S) = 1$.
Axiom 3: If $A_1, A_2, A_3, \ldots$ form a sequence of pairwise mutually exclusive events in S (that is, $A_i \cap A_j = \emptyset$ if $i \neq j$), then

$$P(A_1 \cup A_2 \cup A_3 \cup \cdots) = \sum_{i=1}^{\infty} P(A_i).$$

Figure 6.1: Definition of probability - [68] (p. 30).

Comparing with Kolmogorov's axioms which were introduced in the previous chapter of this thesis, the authors don't present the axioms related to a field but they present the countable additivity as a third axiom. In Kolmogorov's axiomatization, he presents the continuity of a probability measure, which can be shown to be equivalent to countable additivity. The authors mention that if the space is finite, the events A_i go from A_1 to A_n and countable additivity can be replaced by finite additivity.

In what follows, they make an important statement: "*Notice that the definition states only the conditions an assignment of probabilities must satisfy; it does not tell us how to assign specific probabilities to events*" (p. 30). To show how to assign probabilities, they give an example of a coin that is tossed 1000 times and has yielded 800 heads. The use of relative frequencies is justified when they say that we could assign probability 0.5 to each outcome without contradicting the axiomatic definition, but 0.8 to heads and 0.2 to tails is more reasonable and is also in agreement with the axioms.

They use the example of finite sample space, a die toss, to illustrate the axioms 2 and 3. After the example, they explain: "*For discrete sample spaces, it suffices to assign probabilities to each simple event*" (p. 30). This statement is true for a probability space with finite cardinality, but it is not necessarily true for a countable infinite one. This is a first hint that the focus of the book – its discussions and exercises – is exclusively on finite sample spaces; but the lack of making this explicit may result in students not ever questioning or reflecting in the exact issues that provoked

the shift from the classical to the modern approach.

After the axiomatic definition of probability and the short illustrations mentioned above, the book presents an example that works with a finite sample space and asks to assign probability and verify whether the definition satisfies the axioms.

The exercises

We analyzed the 55 exercises proposed by the book around the definition of probability and key concepts. If the sample space size was relevant to solve an exercise, it was a finite one. There is one exercise that asks to describe an infinite sample space, but, for the sake of clarity, it is not contemplated in the Venn diagram. As shown in figure (6.2), out of 55 exercises, 33 may reinforce the association of probability with proportional reasoning. Five exercises are exclusively set operation exercises and only require a counting strategy. 14 exercises require the use of the concepts of independence and conditional probability and require some set manipulation. Only one requires some thinking about the definition of probability. We've found two exercises related to conditional probability that are very interesting because they show that a proportional reasoning may fail to attribute probability to some events. More specifically, when considering a sequence of mutually independent events, each of them with probability p, these two exercises demonstrate that proportional reasoning doesn't work when we want to find the probability of a sequence of n successes of that event, as the probability is p^n and not np.

6.3.2 Discussion

Despite emphasizing the importance of the theory in the preface, the treatment of probability is rather intuitive and informal. In this book, they explain that a probability must satisfy the three axioms, but one interesting fact is that it doesn't specify a method of assigning probabilities to events. Even though the book adopts a frequentist approach, it opens the possibility for other interpretations, or ways of assigning probability to events as long as the assignments satisfy the three axioms.

Another positive attribute of this book is the authors' attempt to guide the students' intuition, showing that the relative frequency model fits in the requirements of the axiomatic probability model when they explain that the relative frequency is non-negative, equals one if you consider the whole space and is finite additive. That is the axiomatic idea of the definition of probability

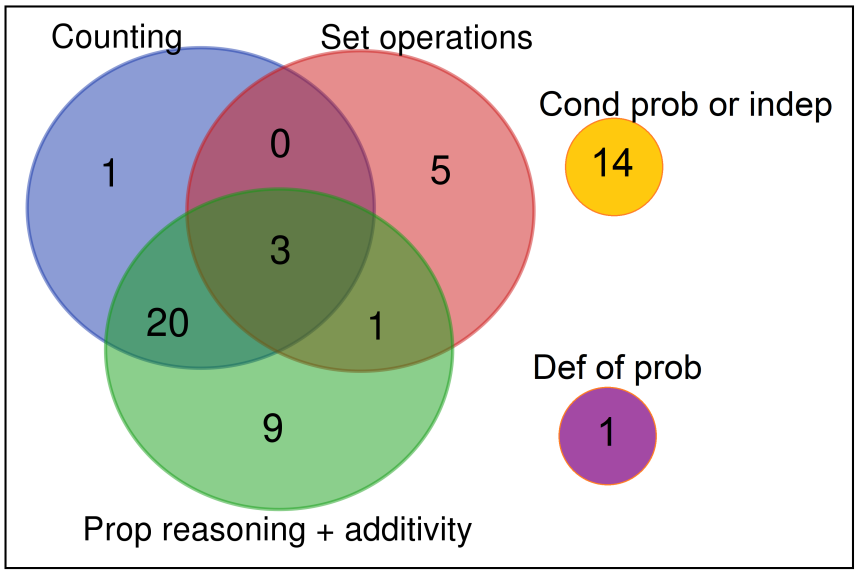


Figure 6.2: Venn diagram - Exercises from [68].

when finite probability spaces are concerned.

However, many important ideas are introduced, but left isolated as they are not treated in the examples or the exercises. To illustrate this, the book takes care to mention that discrete sample spaces are those that have countably many events, however neglects to include any examples or exercises involving probability in spaces which are not finite. The book refers generally to discrete sample spaces, and provides only examples of finite spaces. The unaware reader (the student) may identify discrete with finite. Examples, exercises and claims (such as stating that for discrete sample spaces it suffices to assign a probability to each simple event) may contribute to this identification. This promotes a classical approach in which the axiomatic definition has no raison d'être.

Another problem with the textual part of the chapter is the statement about relative frequency; that it is sufficient to contemplate all real world situations, which is not only false, but also reduces the importance of the axiomatic approach taken by Kolmogorov. We are not saying here that the frequentist approach to probability is not true or relevant to probability. It actually is the most common approach that even Kolmogorov himself used as his interpretation of probability as mentioned in chapter five of this thesis and also in Kolmogorov's book. The problem is when the authors mention that relative frequency is able to handle all probabilistic situations of the real world, they restrict the scope of probability, they take out the development of martingales that were developed from a critique to the limitation of the frequentist approach and mistakenly assign

the credit to von Mises as the founder of modern probability, despite referencing Kolmogorov's axioms in their definition of probability.

The book also states that the relative frequentist approach is sufficient to contemplate all real world situations, which reduces the importance of the axiomatic approach taken by Kolmogorov. We are not saying here that the frequentist approach to probability is not relevant to probability. It actually is the most common approach that even Kolmogorov himself used as his interpretation of probability. The problem we identify is with claiming that relative frequency is able to handle all probabilistic situations of the real world. In doing so, they restrict the scope of probability, taking out, for example, the development of martingales, which were developed from a critique to the limitation of the frequentist approach.

The exercises also seem to exclusively advance a classical approach because they (only) require the use of finite additivity, set manipulation, counting techniques and associate probability to a proportional reasoning. The obstacle of equiprobability and the illusion of linearity may be enhanced by this approach.

6.3.3 Book 2: Ross, S. A first course in probability [52].

The preface informs that the book is intended as an elementary introduction to the theory of probability for students in mathematics, statistics, engineering, and the sciences with the prerequisite knowledge of elementary calculus.

The context

The most important chapter for the goal of this thesis is the second one, where the definition of probability is introduced. After a brief outline of the chapter in the first section, the second section defines sample spaces and events. A sample space is defined as the set of all possible outcomes from an experiment and events are defined as subsets of the sample space. The book presents four examples of finite sample spaces and one uncountable space and also one example of an event from each of the sample spaces. The third section: "*Axioms of Probability*" is the one of interest.

The author begins with an intuition of probability as the limit of relative frequencies of an event from an experiment that was repeated under the same conditions. In what follows he points out of a drawback from that approach:

"*Although the preceding definition is certainly intuitively pleasing and should always be kept in*

mind by the reader, it possesses a serious drawback: How do we know that $n(E)/n$ will converge to some constant limiting value that will be the same for each possible sequence of repetitions of the experiment? For example, suppose that the experiment to be repeatedly performed consists of flipping a coin. How do we know that the proportion of heads obtained in the first n flips will converge to some value as n gets large? Also, even if it does converge to some value, how do we know that, if the experiment is performed a second time, we shall obtain the same limiting proportion of heads? Proponents of the relative frequency definition of probability usually answer this objection by stating that the convergence of $n(E)/n$ to a constant limiting value is an assumption, or an axiom, of the system. However, to assume that $n(E)/n$ will necessarily converge to some constant value seems to be an extraordinarily complicated assumption. For, although we might indeed hope that such a constant limiting frequency exists, it does not at all seem to be a priori evident that this need be the case. In fact, would it not be more reasonable to assume a set of simpler and more self-evident axioms about probability and then attempt to prove that such a constant limiting frequency does in some sense exist? The latter approach is the modern axiomatic approach to probability theory that we shall adopt in this text" (p. 27).

This discussion is very relevant because it points out the limitations of the relative frequency approach and shows the importance of the axioms of probability. After developing an intuition in probability and discussing the limitations of assuming the existence of a limit of relative frequencies as well as mentioning what modern probability is, the author presents 3 axioms of probability in the figure (6.3).

Consider an experiment whose sample space is S. For each event E of the sample space S, we assume that a number $P(E)$ is defined and satisfies the following three axioms:

Axiom 1: $0 \leq P(E) \leq 1$

Axiom 2: $P(S) = 1$

Axiom 3: For any sequence of mutually exclusive events $E_1, E_2, \ldots$, (that is, events for which $E_1 E_j = \emptyset$ when $i \neq j$),

$$P\left(\cup_{t=1}^{\infty} E_t\right) = \sum_{t=1}^{\infty} P(E_t)$$

Where we refer to $P(E)$ as the probability of the event E.

Figure 6.3: Definition of probability - [52] (p. 27)

Following the definition of probability and the discussion of the axioms, the book presents two well-known examples from secondary school: a single throw of a coin and a single throw of a die.

The author summarizes the concept of modern probability: "*The assumption of the existence of a set function P, defined on the events of a sample space S and satisfying Axioms 1, 2, and 3, constitutes the modern mathematical approach to probability theory*" (p. 28). However, there is no definition of set function in the book.

The author doesn't show any agreement between the two examples and the axioms of probability. Instead, he suggests the student try to identify or think about it. Before going to the following section, the author makes a remark about measurable sets: "*We have supposed that P(E) is defined for all the events E of the sample space. Actually, when the sample space is an uncountably infinite set, P(E) is defined only for a class of events called measurable. However, this restriction need not concern us, as all events of any practical interest are measurable*" (p. 29).

In the next section, (2.4), the author uses set operations to develop three properties that are consequences of the axioms:

(1) $P(E^c) = 1 - P(E)$;

(2) If $E \subset F$, then $P(E) \leq P(F)$;

(3) $P(E \cup F) = P(E) + P(F) - P(E \cap F)$.

The author proves each of them and follows with examples that involve set operations and combinatorial techniques from the first chapter of the book. The section that follows (2.5) is all about finite sample spaces with equally likely cases, that is, the classical approach. It presents 15 examples involving different techniques to find probability of specific events, but none are related to the differences between the classical and the modern approaches to probability. Indeed, it is a section based exclusively on classical probability.

Section 2.6, *Probability as a continuous set function*, discusses the continuity of the probability measure. Despite the title, the section doesn't make any analogy with the definition of continuous function from $\mathbb{R}$ to $\mathbb{R}$ that students may be familiar with from a calculus course and the term set function is not defined. In fact, this section is set as an optional part of the book and is the one that deals with infinite probability spaces. It shows the continuity of a probability measure, that is, if we have an increasing or decreasing sequence of events (E_n), then $\lim_{n\to\infty} P(E_n) = P(\lim_{n\to\infty} E_n)$.

This section is set as an optional part of the book and is the one that deals with infinite probability space. It shows the continuity of a probability measure, that is, if we have an increasing or decreasing sequence of events (E_n), then $\lim_{n\to\infty} P(E_n) = P(\lim_{n\to\infty} E_n)$.

There is a nice example in Figure (6.4) of a paradox in probability and the solution uses the continuity of the probability measure and an infinite space.

EXAMPLE 6a Probability and a paradox

Suppose that we possess an infinitely large urn and an infinite collection of balls labeled ball number 1, number 2, number 3, and so on. Consider an experiment performed as follows: At 1 minute to 12 P.M., balls numbered 1 through 10 are placed in the urn and ball number 10 is withdrawn. (Assume that the withdrawal takes no time.) At $\frac{1}{2}$ minute to 12 P.M., balls numbered 11 through 20 are placed in the urn and ball number 20 is withdrawn. At $\frac{1}{4}$ minute to 12 P.M., balls numbered 21 through 30 are placed in the urn and ball number 30 is withdrawn. At $\frac{1}{8}$ minute to 12 P.M., and so on. The question of interest is, How many balls are in the urn at 12 P.M.?

Figure 6.4: Example 6a - [52] (p. 46).

At 12PM there are infinitely many balls because only the balls numbered as $10n$ will have been withdrawn from the urn. Now the experiment is modified as follows:

However, let us now change the experiment and suppose that at 1 minute to 12 P.M. balls numbered 1 through 10 are placed in the urn and ball number 1 is withdrawn; at $\frac{1}{2}$ minute to 12 P.M., balls numbered 11 through 20 are placed in the urn and ball number 2 is withdrawn; at $\frac{1}{4}$ minute to 12 P.M., balls numbered 21 through 30 are placed in the urn and ball number 3 is withdrawn; at $\frac{1}{8}$ minute to 12 P.M., balls numbered 31 through 40 are placed in the urn and ball number 4 is withdrawn, and so on. For this new experiment, how many balls are in the urn at 12 P.M.?

Figure 6.5: Continuation of example 6a - [52] (p. 46).

The ball number n will have been withdrawn from the urn at $\left(\frac{1}{2}\right)^{n-1}$ minutes before 12PM. To conclude, any ball n won't be at the urn at 12PM, so the urn will be empty. Here we can see that the way the balls are taken makes a difference to the result. What happens when the ball to be withdrawn is randomly chosen from the balls in the urn? Using the continuity of probability the measure, it's shown that the urn will be empty at 12PM with probability 1.

This section brings the important and interesting result of the continuity of the probability measure. The example is also interesting because it gives a surprising result and deals with an infinite sample space. It is unfortunate that this section is left as an optional topic, the exercise is interesting, but there is no discussion relating it to the modern approach. It would also be very interesting if a relationship between the continuity of probability and countable additivity, which

is a key fact for modern probability, had been discussed.

The exercises

The total number of exercises analyzed was 97. Out of this total, 20 didn't require any knowledge of the sample space size be resolved. 71 used the information of a finite sample space, five used an infinite sample space and one was related to a finite sample space in the first item and in the second item it changed the sample space to an infinite one. So this specific exercise (problem #45) is mentioned in the both Venn diagrams, for the finite and for the infinite sample space.

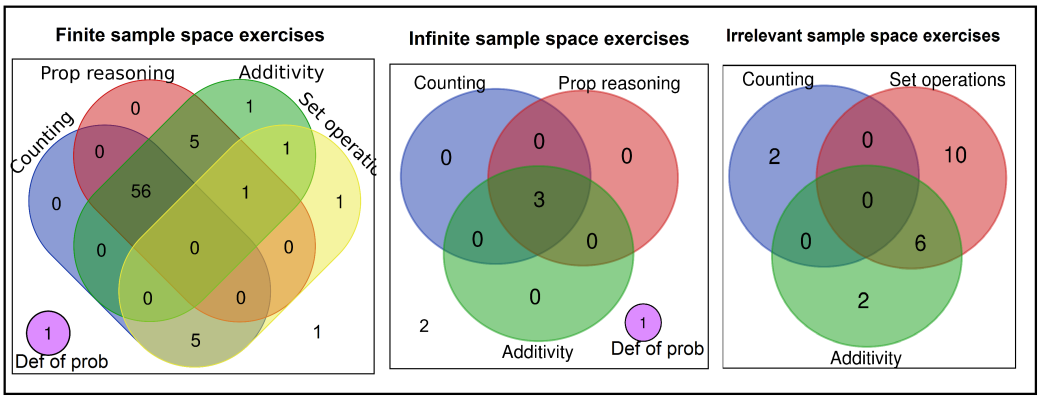


Figure 6.6: Venn diagram - Exercises from [52].

As shown in the Figure (6.6), the great majority of the exercises, 82 out of 97, involved counting techniques and/or related probability with a proportional reasoning and/or the idea of additivity. 24 exercises involved set operations sometimes combined with another category. Three exercises asked to describe the sample space and only two exercises were related to the axiomatic definition of probability.

6.3.4 Discussion

In the book of Ross, we've identified a few "advances" in relation to the one of Wackerly (and others) [68]. The author provides an intuition towards the limit of relative frequency as an interpretation of probability. Then he shows caution and justifies the axiomatization because that limit might not always exists. Another instructive development that he presents is a discussion on axiom three, which is countable additivity. The author takes care to distinguish finite from infinite sample spaces, and shows how countable additivity works in each of these cases.

However, when it comes to the examples, the book is limited to the classics coin toss and die throw. The book doesn't mention how the examples satisfy the axioms, but at least it invites the students to think about it, encouraging them to make this connection for themselves. It then follows a section entirely dedicated to equally likely cases with 15 examples that may result in a very strong enhancement of the classical approach.

The treatment of modern probability is very limited and the classical approach is reinforced. Even though it mentions the concept of modern probability as a set function, the term set function is not defined at all and there are no examples to illustrate any differences between the classical and the modern approaches to probability.

There is a very interesting section that shows probability as a continuous set function and presents an example that uses countable additivity which has surprising results. The down side of it is that this section is set as an optional section and there are no exercises and no relationship between continuity of probability and countable additivity.

The great majority of the exercises makes the students think about proportional reasoning, counting techniques, additivity and set operations on a finite sample space. The axiomatic definition of probability is only approached in two exercises, one in a finite sample space and another one with an infinite sample space. The infinite sample space exercises formed a minority subset of 6 elements. Two simply required the student to describe a sample space, two involved a geometric sum that converges to 1, which in a certain way is also proportional reasoning. There are two exercises that bring attention to a modern perspective of probability. One is an exercise on a finite sample space that is extended to an infinite one. This is very interesting because it makes the student consider the difference between them. The other exercise that makes students think about modern probability is related to the axiomatic definition in an infinite space. Out of 97 exercises, six are related to infinite sample spaces and only two of them call attention to some aspects of modern probability. We take this as evidence of an intensification of the classical reasoning, even though it may call the attention for the existence of a modern approach.

6.3.5 Book 3: Grimmet, G. R. and Stirzaker, D. R. Probability and Random Processes [27].

According to the preface, the book is intended for students at the undergraduate and graduate levels and for those working with applied and theoretical probability. It aims to give a rigorous introduction to probability theory using a limited amount of measure theory, include non-routinely taught topics to undergraduate students and give a 'flavour' of more advanced work in probability.

The context

In this book, our interest lies in the first chapter: *Events and Their Probabilities*. The author introduces the definition of sample space as the set of all possible outcomes of an experiment and events as subsets of the sample space. Two trivial examples of sample spaces are presented: the outcomes of a coin toss and the outcomes of a die throw. The interesting part in this introduction is that it mentions that not all subsets of the sample space are events. The authors say that the set of events is called a field and it is a sub-collection $\mathfrak{F}$ of subsets of the sample space and then it introduces its formal definition. The book follows with the example of a coin being tossed infinitely many times, which configures an infinite probability space, and a σ-field is formally defined. He presents 3 examples of sigma fields and the third one is the σ-field of all subsets of the sample space. He mentions that when the space is infinite, we can't assign probability to all its members, but does not provide any further details on that. The third section introduces the definition of probability. It starts with a frequentist intuition and it mentions, but doesn't explain why, this approach may fail. In the author's own words, "*... writing $N(A)$ for the number of occurrences of [the event] A in N trials, the ratio $N(A)/N$ appears to converge to a constant limit as N increases. We can think of the ultimate value of this ratio as being the probability $P(A)$ that A occurs on any particular trial; it may happen that the empirical ratio does not behave in a coherent manner and our intuition fails us at this level, but we shall not discuss this here*" (p. 4). In what follows, the authors introduce the axiomatic definition of probability as in figure (6.7):

The book also says that a probability measure is a particular case of a measure. It defines measure as a countably additive function $\mu : \mathcal{F} \to [0, \infty)$ satisfying $\mu(\emptyset) = 0$. If $\mu(\Omega) = 1$, μ is a probability measure. Then the examples of the coin toss and die throw are presented again, followed by some properties that derive from the axioms:

The continuity of the probability measure is presented as a lemma which is half proven and

(1) Definition. A **probability measure** $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is a function $\mathbb{P} : \mathcal{F} \to [0, 1]$ satisfying

(a) $\mathbb{P}(\varnothing) = 0, \quad \mathbb{P}(\Omega) = 1;$

(b) if $A_1, A_2, \ldots$ is a collection of disjoint members of $\mathcal{F}$, in that $A_i \cap A_j = \varnothing$ for all pairs i, j satisfying $i \neq j$, then

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

The triple $(\Omega, \mathcal{F}, \mathbb{P})$, comprising a set Ω, a σ-field $\mathcal{F}$ of subsets of Ω, and a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$, is called a **probability space**.

Figure 6.7: Definition of probability - [27] (p. 5).

(4) Lemma.

(a) $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$,

(b) *if* $B \supseteq A$ *then* $\mathbb{P}(B) = \mathbb{P}(A) + \mathbb{P}(B \setminus A) \geq \mathbb{P}(A)$,

(c) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$,

(d) *more generally, if* $A_1, A_2, \ldots, A_n$ *are events, then*

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i} \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j) + \sum_{i<j<k} \mathbb{P}(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n+1} \mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_n)$$

where, for example, $\sum_{i<j}$ *sums over all unordered pairs* (i, j) *with* $i \neq j$.

Figure 6.8: Lemma - [27] (p. 6).

half left as an exercise. The book also mentions that the continuity of the probability function is equivalent to countable (instead of finite) additivity of P. To finish the section, the authors mention that null events (or probability 0 events) are not impossible. They give an intuitive example of the probability that a dart strikes any given point of the target is 0, but not impossible. This discussion is very interesting and was pointed out by Poincaré, as presented in chapter four of this thesis.

The exercises

The chapter has 46 exercises and the diagram A in the figure (6.9) contain 44 of them. For the sake of simplicity and comprehensibility of it, diagram A contemplates four different categories. Exercise number 2 (which asks the students to describe an infinite sample space) and exercise number 7 (which requires exclusively the concept of independence) do not fit those categories and are not contemplated in diagram A. The diagram B contains the exercise number 2 and also details some exercises from diagram A that are more interesting from the point of view of this thesis.

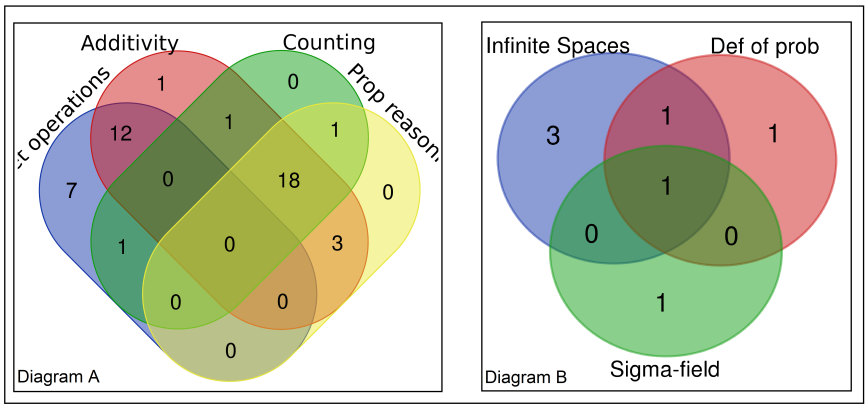


Figure 6.9: Venn diagram - Exercises from [27].

Diagram B shows how many exercises and problems require students to review the definition of probability and the concepts of a sigma-field and infinite sample spaces.

Most of the exercises in this book are about proportional reasoning, counting techniques, additivity and set operations in a finite sample space. Only five exercises involve an infinite sample space and only three make the students think about the axiomatic definition of probability. Even the idea of a sigma-field is only visited in two exercises. Out of 46 exercises, only 7 give a significant contribution towards a modern approach.

6.3.6 Discussion

Grimmet and Stirzaker's book has a more sophisticated mathematical treatment than those by Ross[52] and Wackerly (and others)[68]. It can be used in undergraduate as well as in graduate courses, but doesn't require any knowledge of measure theory. It introduces the concepts of algebra, σ-algebra and measure, but doesn't compare them or present the Caratheodory's extension theorem. They have ventured further into a modern approach than the previously discussed texts, but there is no discussion or comment that calls attention to the differences between the classical and the modern approach. The book mentions some interesting ideas connected to modern probability but doesn't justify or exemplify them. To illustrate that, we can remark i) the existence of non-measurable sets, ii) the fact that in infinite spaces, we can't assign probability to every simple event and iii) that the limit of relative frequencies may not exists, but no examples on these three items are given. The examples are usually the classical coin toss and die throw, however we need to remark the one example about an infinite sequence of a coin toss which calls attention to the

need for a σ-algebra, which is more interesting in the point of view of modern probability.

There is a theorem that proves the continuity of the probability measure and one exercise related to that continuity, but no discussion on the definitions or implications of continuous set functions or continuity of measure is presented in the text. The relationship between countable additivity and the continuity of probability is not discussed. In a total of 46 exercises, only four require a reasoning about an infinite sample space and three exercises are about the axiomatic definition of probability. Overall, the book touches some important facts that could lead to a more precise understanding of modern probability, by confronting the epistemological obstacles of equiprobability and proportionality, and addressing the confusion with the classical approach, however there is no discussion that calls attention to those aspects and that important distinction is left aside.

6.3.7 Book 4: Shiryaev, Probability 1 [59].

This book is from a collection called *Graduate Text in Mathematics*. It is divided into eight chapters and the author advises in the preface that it should be taught in three semesters. In the introduction, the author says that the book "*is based on Kolmogorov's axiomatic approach. However, to prevent formalities and logical subtleties from obscuring the intuitive ideas, our exposition begins with the elementary theory of probability, whose elementariness is merely that in the corresponding probabilistic models we consider only experiments with finitely many outcomes. Thereafter we present the foundations of probability theory in their most general form*" (p. xvi). This is the same approach adopted by Kolmogorov in his book, where the first chapter deals with finite probability spaces and the second one generalizes it to infinite spaces. Therefore, we will discuss two parts of Shiryaev's book: the first section of chapter one, where the definition of probability is introduced considering finite probability spaces and the first section of chapter two, where the definition is presented in the most general form.

The context

The first section of chapter one starts with the definition of a finite sample space with the basic examples of a coin toss and a die throw. It also presents other finite sample spaces that are illustrated with combinatorial techniques. The concept of event is introduced as "*all subsets A of Ω for which, under the conditions of the experiment, it is possible to say either 'the outcome $\omega \in A$'*

or 'the outcome $\omega \notin A$*'* "(p. 6).

To clarify the concept of event, the author presents a counter-example with a non-measurable set, which cannot be an event, using the sample space of an experiment of three tosses of a coin. He calls the set $A = \{HHH, HHT, HTH, THH\}$, the event of appearance of at least two heads. If we can determine only the result of the first toss, this set A cannot be an event, since we cannot say if an outcome ω is an element of A or not. Then the author introduces the formal definition of an Algebra of events.

He presents three examples of algebras: i) the trivial algebra, ii) the algebra generated by A and iii) the algebra of all subsets of Ω. He then presents the concept of decomposition, $\mathfrak{D}$, and says that all unions of the sets in a decomposition form an algebra, called the algebra induced by the decomposition $\mathfrak{D}$. As an example, he shows the decompositions in each of the three examples of algebra. The decomposition will be used later in this book to treat probabilities conditional to a decomposition. This step develops a nice intuition that facilitates the understanding of probabilities conditional to a σ-algebra.

In what follows, probability is defined for a finite sample space as a weight, $p(\omega_i)$, that is assigned to each outcome $\omega_i \in \Omega, \quad i = 1, \ldots, N$ and has the following properties:

(1) $0 \leq p(\omega_i) \leq 1$;

(2) $p(\omega_1) + \cdots + p(\omega_N) = 1$.

The probability $P(A)$ of any event $A \in \mathcal{A}$ is defined as $P(A) = \sum_{i:\omega_i \in A} p(\omega_i)$, where $p(\omega_i)$ is the 'weight' of the outcomes ω_i.

In the next step, the author presents the triple $(\Omega, \mathcal{A}, P)$ as a 'probability space' or a 'probabilistic model' of an experiment with a finite space of outcomes Ω, and algebra of events $\mathcal{A}$ and follows with five properties developed from the definition:

(1) $P(\emptyset) = 0$;

(2) $P(\Omega) = 1$;

(3) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$;

(4) If $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$;

(5) $P(\bar{A}) = 1 - P(A)$.

In subsection 5 he explains that the assignment of $p(\omega_i)$ can be done, to a certain extent, by relative frequencies or by considering all outcomes equally probable, which is the classical method. He then finishes this section with two examples that uses combinatorial techniques. It's worthy to note that there is an emphasis that the space is finite, $\mathcal{A}$ is an algebra and P is not called a measure in this chapter.

We now move to chapter two, "Mathematical Foundations of Probability Theory", where he presents the definition of probability in full generality, instead of that for finite spaces. The first section is called: "Probabilistic Model for an Experiment with Infinitely Many Outcomes: Kolmogorov's Axioms".

It starts with the construction of a probabilistic model for the experiment of an infinite number of independent tosses of a coin. The sample space is composed by the sequences $\omega = (a_1, a_2, \ldots)$ whose elements are 0 or 1. He says that there is a one-to-one correspondence between the points ω of Ω and the points a of the interval $[0, 1)$, so Ω has the cardinality of the continuum. This is immediately followed by an explanation that we can't assign probabilities using symmetry, like in the classical method, nor assign probabilities to individual outcomes from uncountable sample spaces. It is worth putting this explanation in the author's own words, as can be seen in figure 6.10.

Since we may take Ω to be the set $[0, 1)$, our problem can be considered as the problem of choosing points at random from this set. For reasons of symmetry, it is clear that all outcomes ought to be equiprobable. But the set $[0, 1)$ is uncountable, and if we suppose that its probability is 1, then it follows that the probability $p(\omega)$ of each outcome certainly must equal zero. However, this assignment of probabilities $(p(\omega) = 0,\ \omega \in [0, 1))$ does not lead very far. The fact is that we are ordinarily not interested in the probability of one outcome or another, but in the probability that the result of the experiment is in one or another specified set A of outcomes (an event). In elementary probability theory we use the probabilities $p(\omega)$ to find the probability $\mathsf{P}(A)$ of the event A: $\mathsf{P}(A) = \sum_{\omega \in A} p(\omega)$. In the present case, with $p(\omega) = 0,\ \omega \in [0, 1)$, we cannot define, for example, the probability that a point chosen at random from $[0, 1)$ belongs to the set $[0, \frac{1}{2})$. At the same time, it is intuitively clear that this probability should be $\frac{1}{2}$.

These remarks should suggest that in constructing probabilistic models for uncountable spaces Ω we must assign probabilities not to individual outcomes but to subsets of Ω.

Figure 6.10: Assigning probability to infinite sets - [59] (p. 160).

At this point he introduces some definitions to prepare the student to move to the modern

approach. He defines: i) an **algebra** of subsets of Ω, ii) a **finitely additive measure on an algebra** as a finite additive set function $\mu = \mu(A)$ that gives non-negative values when applied to the subsets $A \in \mathcal{A}$ and iii) a **finitely additive probability measure** when $\mu(\Omega) = 1$.

In the second subsection, Shiryaev recalls the definition of a probabilistic model, but this time with an infinite set Ω. He says that this model is "*too broad to lead to a fruitful mathematical theory*" and the class of subsets of Ω and the class of probability measures must be restricted. This fact justifies the definition of a σ-algebra, a measurable space and a countably additive measure (or simply measure). Then he says that a measure on a σ-algebra that satisfies $P(\Omega) = 1$ is called a **probability measure**.

The book then presents some properties that arise from the definition of probability in an infinite sample space, just like in chapter one for the finite case. To finish the section, the author presents the theorem that shows that countable additivity is equivalent to continuity of the probability measure and defines a probability space according to Kolmogorov's axiom.

The exercises

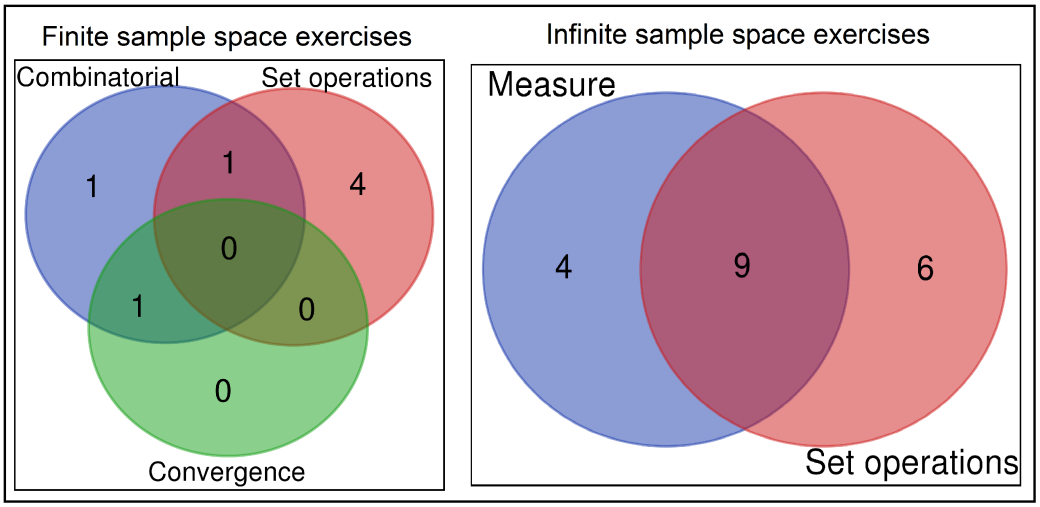


Figure 6.11: Venn diagram - Exercises from [59].

As we can see in the figure (6.11), the exercises from this book are primarily tasks of combinatorial analysis and set operations in a finite sample space and when the sample space is infinite, the tasks form a mix of measure theory, set operations or some combination of the two. There are no exercises that require some thinking of the axiomatic definition or the changes from the

classical to the modern approach to probability.

6.3.8 Discussion

Shiryaev's book, just like Kolmogorov's [39], separates finite and infinite samples spaces into two different chapters, which clearly establish a difference in the treatment of probability between them. The first chapter formalizes the classical probability with the axioms using an algebra of sets, which prepares the students for the language and notation in the second chapter. It also gives an example of non-measurable events. The second chapter starts with an example in an infinite space that shows the limitation of the classical approach and motivates the axiomatic definition using a σ-algebra. In this example it also mentions that for uncountable spaces we can't assign probabilities to individual outcomes, so we must take subsets of the sample space. He ends the section showing the connection between countable additivity and the continuity of the probability measure.

The theoretical part of the book presents a formal definition of probability and develops it in a way that demonstrates the difference between the classical approach and the modern one. It shows the connection between the continuity of probability and countable additivity, and that for uncountable sample spaces we can't attribute a positive probability to each individual point of the set. The narrative proposes a nice path from classical to modern probability. The drawback of the book is the lack of examples and exercises illustrating such path or the differences between the two approaches. In the second chapter, the exercises can be mostly solved with results from measure theory and set operation and require very little knowledge of probability.

6.3.9 Book 5: Durrett, R. Probability: Theory and Examples [22].

The preface of the book mentions: i) that the book focuses on examples for people who apply probability in their work and ii) that the book contains exercises because probability is not a spectator sport. Despite the preface mentioning that the book is for people who use probability in their daily work, it is deeply theoretical and almost no applied situations can be found. There is no explicit need for a previous measure theory course because the topics of that discipline are given along with the probability content. However, we wonder if given the level of abstraction and complexity of the mathematical reasoning required in this book, a previous measure theory course should be recommended, to allow students to concentrate their efforts on the topic at hand,

probability.

The context

The book doesn't present any introductory discussion, analogy or comparison to develop an intuition for probability. In the first page of chapter one, the book goes straight to the definition of a probability space with the definitions of a σ-field, a measurable space, a measure and a probability measure as in Figure (6.12).

A **probability space** is a triple $(\Omega, \mathcal{F}, P)$ where Ω is a set of "outcomes," $\mathcal{F}$ is a set of "events," and $P : \mathcal{F} \to [0, 1]$ is a function that assigns probabilities to events. We assume that $\mathcal{F}$ is a **σ-field** (or **σ-algebra**), i.e., a (nonempty) collection of subsets of Ω that satisfy

(i) if $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$, and

(ii) if $A_i \in \mathcal{F}$ is a countable sequence of sets then $\cup_i A_i \in \mathcal{F}$.

(...)

Without P, $(\Omega, \mathcal{F})$ is called a **measurable space**, i.e., it is a space on which we can put a measure. A **measure** is a nonnegative countably additive set function; that is, a function $\mu : \mathcal{F} \to \mathbf{R}$ with

(i) $\mu(A) \geq \mu(\emptyset) = 0$ for all $A \in \mathcal{F}$, and

(ii) if $A_i \in \mathcal{F}$ is a countable sequence of disjoint sets, then

$$\mu(\cup_i A_i) = \sum_i \mu(A_i)$$

If $\mu(\Omega) = 1$, we call μ a **probability measure**. In this book, probability measures are usually denoted by P.

Figure 6.12: Definition of probability space - [22] (p. 1)

Following these definitions the author presents the properties of: i) monotonicity, ii) subadditivity, iii) continuity from below and iv) from above of a measure. The first example is the classic die throw (finite space), but with a die that is not necessarily balanced. There is one exercise that asks to show that an arbitrary intersection of σ-algebras is also a σ-algebra, and the smallest σ-algebra containing $\mathcal{A}$ is the σ-algebra generated by $\mathcal{A}$.

The book gives an example of a measure on the real line, a Stieltjes measure function, and defines a semi-algebra with an example of semi-open intervals in $\mathbb{R}^d$. It then introduces the definition of an Algebra and gives an example of a collection that is an Algebra, but not a σ-algebra. A lemma showing that finite disjoint unions of sets in a semi-algebra form an algebra is presented,

followed by an example of such an extension.

The author then defines a measure on an algebra and presents a theorem (with the proof in the appendix) that helps to extend a measure on a semi-algebra to the σ-algebra it generates. The rest of the section lies outside our interest for this thesis because it extends the definition of a probability measure on $\mathbb{R}$ to $\mathbb{R}^n$.

The exercises

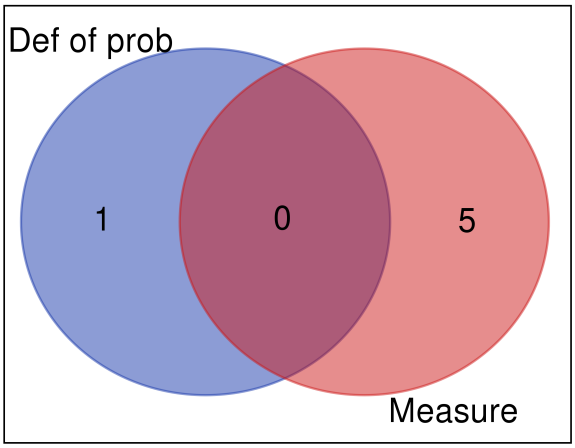


Figure 6.13: Venn diagram - Exercises from [22]

Durrett's book is the one with the smallest set of exercises. Out of six exercises from the section, one makes the students work with the axiomatic definition of probability and the other five work exclusively with measure theory. Among those measure theory exercises, two concern the definition of an algebra and three the generation of a σ-algebra.

6.3.10 Discussion

The book of Durrett is very direct and introduces no context for the definition of probability. It states the definition of probability followed by some properties and one general example of a discrete probability space (finite and infinite). There is one line where he says: "*In many cases when* Ω *is a finite set, we have* $p(\omega) = 1/|\Omega|$", but this shy statement is not illustrated throughout the text, except in that phrase. The less attentive, or unaware reader may not notice the importance of that restriction to the cases when Ω is finite. Another characteristic of this book is that is doesn't require a previous course in measure theory. Concepts such as σ-algebra, semi-algebra

and measure are introduced along with the probability ideas. Most of the analyzed exercises focus on measure theory and the ideas purely in probability seem to be of secondary importance.

6.4 Final remarks

While exercises in textbooks may be tools for students to practice a technique, if understanding requires confronting and surmounting epistemological obstacles, exercises should help students confront their beliefs and previous inadequate knowledge. They should help students testing whether they have grasped the *right* idea of a specific definition, theorem, property or if they are aware of subtle consequences of the mathematical content in focus. Exercises also help students becoming aware of which ideas are the most important. In other words, exercises give clues to students of what is the knowledge they are expected to learn. When students start learning a new subject, it's natural that they don't have the mathematical maturity to distinguish and keep in mind the most important ideas out of a myriad of information and less important facts by simply reading the textbook.

The five books that we've analyzed provide much emphasis on counting techniques, set operations and measure theory facts, to the detriment of the innovations that modern probability brought, such as the axiomatic definition and the ability to handle infinite sample spaces. The undergraduate level books reinforce the classical approach by showing the trivial and classical examples of the coin toss and die throw and also by proposing a great number of exercises that associate proportional reasoning in a finite sample space with probability. The graduate level books show less of a tendency to associate probability to proportional reasoning. On the other hand, they concentrate a great deal of effort on set operations and measure theory exercises, leaving aside the innovations brought by Kolmogorov.

We recognize the value in including well known examples, such as the coin toss or the die throw. Not only will most students recall these scenarios from earlier courses, but this type of tangible real-world event can offer a more palatable introduction to more abstract concepts. Furthermore, we are not denying or neglecting the importance that set operations, counting techniques and measure theory deserve when it comes to probability. However, our analysis points out : i) the limitation of examples and exercises to the very simple cases where the classical approach with the ideas of finite probability spaces and proportional reasoning are reinforced in the undergraduate

level books and, ii) that working almost exclusively with measure and set theory ideas that require little (or occasionally no) knowledge of probability may deviate the focus from the key concepts to be learned in the graduate level books.

The book by Wackerly (and others)[68] has a more intuitive than formal approach. It discusses how the relative frequency fits in with the axioms, however the treatment is concentrated in finite sample space situations. Despite their mention of countable additivity in the third axiom, the situations presented in the book largely reinforce probability as a proportional reasoning in a finite sample space and no importance to the innovations brought by Kolmogorov is given. While Wackerly (and others) reduces the importance of the axioms when they say that all real world situations fit in the relative frequency perspective, Ross[52] justifies the axiomatization by saying that the relative frequency may fail, so we need first to define a set of axioms and then verify that the relative frequency, when the limit exists, fits those axioms. Ross also reinforces the idea of probability as a proportional reasoning in a finite sample space via the exercises as well as through a section with 15 examples of those cases. The section that discusses the continuity of probability is left as an optional topic, doesn't have any exercises and is not related to countable additivity.

Unlike the first two books analyzed, the book of Grimmet and Sirzaker[27] is not meant to be used in an introductory course. It advances with an example that shows the need for a σ-algebra when considering infinite spaces and mentions some topics related to a modern approach, but doesn't detail any of them. Most of the exercises also place an emphasis on proportional reasoning, counting techniques, additivity and set operations, which drives the attention away from the modern approach.

Shiryaev's book [59] presents the distinction between the classical and the modern approach more clearly, and explicitly, than the others. The text presents the definition of an algebra for finite sample spaces and when it extends to infinite spaces in chapter two, there is an example that shows that the approach to probability needs to be changed. It also gives examples of a non-measurable set and connects countable additivity with the continuity of probability. The drawback is the lack of examples in the second chapter and the fact that the exercises that are provided concentrate heavily on measure theory and set operations rather than on the modern definition of probability. Like Shiryaev, Durrett's book [22] doesn't associate probability with a proportional reasoning. It actually introduces many ideas from the modern approach. However, the book is goes directly into the definitions and doesn't present any discussion that would develop any intuition of probability

or any importance of certain concepts and ideas. As this books requires a more abstract and sophisticated reasoning than the previous ones, it is not meant for a first course in probability. A final remark on this book is that the focus on measure theory content may drive the attention away from probability.

As presented in chapter four of this thesis and in Von Plato [67], the real world doesn't possess the symmetries of the classical theory, and the approach presented in the undergraduate analyzed books can be harmful because rather than provoking the reader to confront and overcome the epistemological obstacles of equiprobability and proportionality, they may actually be reinforcing them. In the case of the graduate books we analyzed, the focus may deviate the attention from these obstacles. Hence, the main critiques we present here are that when defining probability, giving examples and proposing exercises, we should go beyond trivial examples and concern ourselves with how much the example and exercises make students work with probability or other mathematical contents. While simple examples set students in a familiar context to understand probability, and measure and set theory are crucial to probability as we've shown in chapter 4, our point is that simple examples and other mathematical contents should be present in the books, but the focus should be changed, to illustrate some of the key aspects introduced by Kolmogorov's work driving the students from the classical towards the modern thinking in probability.

Chapter 7

Final Remarks

The objective of this thesis was to answer two main questions, the first concerning the history and foundation of probability and its association with measure theory. If probability has been present in mathematics for many centuries, why did the advent of measure theory immediately reveal a very strong relationship between these two branches of mathematics, establishing probability as a measure between 0 and 1? More specifically, why did probability *need* measure theory as its basis to be considered an autonomous branch of mathematics? By understanding this evolution from classic to modern probability and the importance of Kolmogorov's axiomatization, a second main research question that attaches a didactical value to this thesis emerged: Considering the classical and the modern approaches to probability, which one of them are primarily advanced by undergraduate and graduate textbooks?

7.1 Remarks on the history and foundation of probability

Bernoulli and de Moivre published the first works that defined probability, starting what we have called in this thesis the classical approach. We consider them the founders of probability as science due to the great level of generality of their definitions of probability and expectation. Despite the advances made by Cardano, Pascal and Huygens among others, their works were limited to resolution of specific problems, while Bernoulli and de Moivre defined probability as a general concept: the ratio of favourable over possible outcomes. Bernoulli also stated and proved, with full rigour, the first version of the weak law of large numbers.

Classical probability was considered a branch of applied mathematics. With its development,

probability provided formulas for error terms, statistical physics and solutions to problems in games of chance, however not much attention was given to the mathematical basis of that probabilistic context. The concepts and methods were specific to applications, and their contributions to larger questions of science and philosophy were limited. The classical definition remained essentially the same throughout the 18^{th} and 19^{th} centuries. Yet, as science evolved through time, the lack of precision of some associated concepts such as random variables and events, and some contradictory results began to evidence the limitations of that definition of probability.

The modern and axiomatic definition of probability in its complete and abstract form could not be developed until the advent of measure theory. The definition of measurable sets broadened the type of sets for which we can evaluate the probability. The Carathéodory's extension theorem or the equivalent probability version by Kolmogorov relied heavily on countable additivity. Lebesgue's integral allowed the proof of many convergence theorems involving limits of integrals. Fréchet took Lebesgue's integral beyond Euclidean spaces and the Radon-Nikodyn theorem developed the integral in full abstraction, providing a way to evaluate the probability of a set conditional to a σ-algebra, which made it possible to see conditional probability as a random variable and also to evaluate probability conditional to sets of measure zero.

The first probability works that used measure theory came form Anders Wiman in 1900 and Weyl in 1909. This relationship was consolidated by Hausorff in 1914. Despite the association between the disciplines, the modern definition of probability was still to be created. Besides Hausdorff's work in set theory, Borel's use of countable additivity, his strong law of large numbers and the different demonstrations from Hardy and Littlewood, Hausdorff, Khinchin and Kolmogorov, made important contributions in that direction.

Rather than proceeding with a purely chronological exposition, we tried, as much as possible, to explore the main ideas, even the blind alleyways, that led to the axiomatization of probability based on measure theory. These imprecise and contradictory developments are important for scientific evolution. The history of mathematics cannot be limited to the results in the standard textbooks, and probability is not an exception. The early and unsuccessful attempts at an axiomatization from Laemmel, Broggi, Hilbert, Lomnicki, Ulam, von Mises, Slutsky and Steinhaus provided important insights to the advance of probability.

Besides the failed attempts at an axiomatization, successful developments were also important.

Daniell's integral of a linear operator with examples in infinite-dimensional spaces, Wiener's formalization of the notion of Brownian motion and Ville's concept of martingales are examples in that direction. Other works from Kolmogorov also made significant progress towards modern probability before the axiomatization. In particular, we note his articles of 1925, on convergence of random events, and 1928, on convergence of the sum of random variables. His article of 1929 improved the association of measure and probability using a countable additive function and his work of 1931 developed continuous time Markov chains using countable additivity.

If one takes into consideration all of those contributions, it is not hard to conclude that Kolmogorov's *Foundations of the Theory of Probability* is a work of synthesis. Kolmogorov was the mathematician who was able to identify the valuable ideas of his predecessors among the myriad of statements to fit existing knowledge into a new approach. To do justice to the scope of his book, beyond the synthesis and the axiomatic definition, there are also a number of innovations that must be taken in consideration: i) probability distributions in infinite-dimensional spaces, ii) differentiation and integration of mathematical expectations with respect to a parameter and iii) a general treatment of conditional probabilities and expectations based on Radon-Nikodyn's theorem that allowed us to evaluate probabilities conditioned to measure zero sets.

As Kolmogorov mentions, his developments arose of necessity from some concrete physical problems. Quantum mechanics viewed the elementary processes in nature as non-deterministic and modern probability played (and still plays) an essential role in describing those processes.

We've exposed how Kolmogorov's book constructed the axiomatization in two chapters of his book. In the first one, he presented five axioms considering a finite sample space. The main contribution there is the set of axioms that formalized and generalized the classical definition in finite spaces. The second chapter added another innovation to the definition, because it reaches its full generality when Kolmogorov introduced axiom VI (continuity [1]) and formalized probability to infinite spaces. We've also described the concepts of probability functions, random variables, conditional probability and conditional mathematical expectation in the modern approach. After the definition of those terms, we used modern probability to resolve the great circle paradox as an illustration of how this approach established a rigorous basis free of ambiguities. The paradox in the great circle arises from a mistaken application of elementary conditional probability, coming

[1] or equivalently, countable additivity

from a naïve application of symmetry[2] that leads to an application of a 1-dimensional Lebesgue measure in a situation where it should be 2-dimensional.

7.2 Remarks on the didactical implications

The understanding of this evolution from the classic to the modern approach to probability made us think about students' understanding of the definition of probability and the approach that books advance through the exposition and the exercises.

After a discussion into the concepts of epistemologial obstacles in mathematics and in probability, we've identified a great scarcity of studies involving probability teaching and learning at a post-secondary level. Many studies have been done concerning elementary or high school students and a great measure of those are dedicated to conditional probability, randomness, and representativeness of samples. However, there is a lot of research to be done in students' understanding of the definition of probability at the post-secondary level.

Given the shortage of research in post-secondary probability education, we've done a pilot test with a few graduate students from mathematics and statistics programs to obtain some insights into their conceptualization of probability. In order to accomplish this, we exposed them to a situation where they needed to deal with an infinite space, where the classical approach is ineffective. We've identified the persistence of the equiprobability and proportionality obstacles in those students. In probability, these epistemological obstacles come from the habit of using proportional reasoning in finite spaces with equally likely events, which reinforces a classical approach to probability.

Once those obstacles were identified, we decided to investigate some undergraduate and graduate textbooks used in the four universities in Montreal with the goal of identifying how those books introduce the definition of probability and help develop, though the exercises, a modern or a classic view of the subject.

The main conclusion that we were able to draw from this analysis, was that too much emphasis is given to counting techniques, set operations and, for the graduate level, measure theory facts, and the innovations that modern probability brought play a secondary role. The undergraduate level books reinforce the classical approach by showing the trivial and classic examples of the coin

[2] or the principle of indifference

toss and die throw as well as by proposing tasks that, for the most part, stimulate an association of probability with proportional reasoning in a finite sample space. The graduate level books don't show the same tendency of attaching a linear reasoning to probability as do the undergraduate ones. They do, however, encourage students to concentrate their efforts on exercises whose content is almost exclusively drawn from set and/or measure theory, many of them not demanding any knowledge of probability at all.

Simple and trivial examples such as the die throw and coin toss don't do harm in and of themselves. In fact, it can be beneficial to consider a classic example to begin with, because it puts the student in a familiar setting that can enhance their confidence for learning probability. Additionally, we don't deny the importance of set operations, counting techniques and measure theory to probability and we don't advocate for the removal of those exercises from the books. Our critiques here addresses to the (almost) exclusiveness of exercises that stimulate a proportional reasoning and an equiprobability view of sets, in a classical probability context, in the case of the undergraduate books, and the focus on almost exclusively measure and set theory tasks with little probability knowledge requested, in the case of the graduate books.

The approach presented in the books analyzed can be harmful, because instead of overcoming the epistemological obstacles of proportionality and equiprobability, they may be actually reinforcing them, in the case of undergraduate level, or deviating the attention from them in the case of the graduate level. When choosing examples and exercises, we suggest that the instructor should consider the weight given to classical probability, modern probability or other mathematical contents. Our main point here is that simple examples and other mathematical concepts should be present in the recommended texts (or in the lectures), but the focus should be changed, to illustrate some aspects introduced by Kolmogorov's work that drive the students from the classical towards the modern thinking in probability.

7.3 Originality, limitations and future research

One of the elements of originality of this thesis is the use of original sources to evidence a few mathematical ideas, or to present in detail some proofs, from each author's contribution to the foundations of modern probability. The didactical contribution is also original, because there is a scarcity of research in the pedagogy pertaining to the teaching and learning of probability at a

post-secondary level.

This thesis is limited to some ideas of the evolution of probability. We did not attempt to outline the entire history of probability theory; detailing every development and every proof is well beyond the scope of this work. Rather, we focused on some main ideas and detailed the proofs of some important results.

Some important original sources were not accessible in English or French. For example, Carathéodory's work on axioms for measure theory or his extension theorem and Hausdorffs book in set theory, with important insights in probability, were only accessible in German[3].

Our pilot test is concentrated with a very small sample extracted from a restricted population. The results achieved in this test were mainly used as preliminary ones to bring insight into further investigation. Instead of the students' conceptualization of probability, the focus of our research was the analysis of the theory and exercises presented in five books typically chosen in the four universities in Montreal for undergraduate and graduate courses in probability.

We surmise that if students at the graduate level in mathematics don't display a modern thinking of probability, undergraduate students, or students from different disciplines will not display this modern thinking either.

In future research, this study could be extended to a larger sample of students and from different areas that are highly connected to probability, such as engineering or computer science. In the case of other disciplines, one may question what is at stake, if anything, if professionals don't develop a modern approach to probability.

Further questions could concern difference between students who have a background in measure theory and those who have none. If differences exists, which elements of measure theory, and what didactic approach, would help students developing a modern approach to probability? In particular, what type of tasks (exercises) would help students reflecting on a modern approach to probability and understanding the importance and shortcomings of the classical approach?

[3]There is an English translation, but it is abbreviated and the chapter that discusses probability has been omitted.

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